Recap: group actions on sets

$$S_X := \{ X \xrightarrow{f} X : f \text{ 1-1 \& onto} \}$$

group action Φ:

$$\Phi : G \to S_X$$

$$g \mapsto \phi(g, x) =: g \cdot x$$

$$\phi : G \times X \to X$$

$$\begin{cases} g_1 \cdot (g_2 \cdot x) = (g_1 g_2) x. \\ 1_G \cdot x = x \\ g \cdot (g^{-1} \cdot x) = x \end{cases}$$

set X + group action by G : G-set

$X = G$.
$$\begin{cases} g_1 \cdot x = g_1 x & x, g \in G = X \\ g \cdot x = g x g^{-1} & x, g \in G \end{cases}$$

$GL(n, K)$ acts on K^n matrix-vector mult.

$\hookrightarrow$ $\{ g \mid z \}$ on $\mathbb{R}^3$

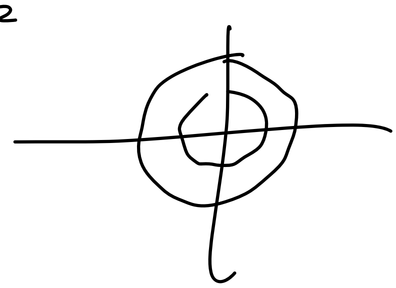
Def: Orbits
$$O_G(x) = \{ y : \exists g \in G \text{ s.t. } y = g \cdot x \}$$
$$:= \{ g \cdot x : g \in G \}$$

$x \sim x'$: $x, x' \in O_G(x)$

X: $\quad O_G(x_1) \cap O_G(x_2) \neq 0 \Rightarrow O_G(x_1) = O_G(x_2)$

set of orbits: X/G.

$SO(2)$ on $\mathbb{R}^2$



Def. equivariant map $\quad f: X \to X'$

$$f(g\cdot x) = g\cdot f(x) \quad \forall x \in X.\ g \in G$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

Induced actions on associated function space

$X, Y.\quad \mathcal{F}[X \to Y]$: set of functions from X to Y

X : group action of $G \quad \phi: G \times X \to X$

$\tilde{\phi}$. action on $\mathcal{F}[X \to Y]$:

$$\tilde{\phi}(g, F)(x) := F(\phi(g^{-1}, x)) \in \mathcal{F}[X \to Y]$$

$$(g\cdot F)(x) = F(g^{-1}x)$$

Example:

G : $\{R | \vec{\tau}\}$ acts on $\mathbb{R}^3$

$\mathcal{F}[\mathbb{R}^3 \to \mathbb{C}] \Rightarrow \varphi(\vec{r})$ wavefunctions

$$(g \cdot F)(gx) = F(x)$$

$$\underline{\underline{(g \cdot F)(x) = F(g^{-1}x)}}$$

left G-action.

right G-action.

$$X \xrightarrow{\phi(g,\cdot)} X$$

$$x \longmapsto x\cdot g$$

$$\begin{cases} (x\cdot g_1)g_2 = x\cdot(g_1 g_2) \\ x\cdot e = x \end{cases}$$

Example: $G = SO(2,\mathbb{R})$ on $\mathbb{R}^2$.

left action:
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$\Rightarrow$

right action:
$$(x, y)\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = (x', y')$$

right $\Leftrightarrow$ left construct using inverse elements of the group

$g^{-1} \Leftrightarrow g$

5. The symmetric group

recall set of permutations:

$$S_X := \{ X \xrightarrow{f} X : f \text{ invertible} \}$$

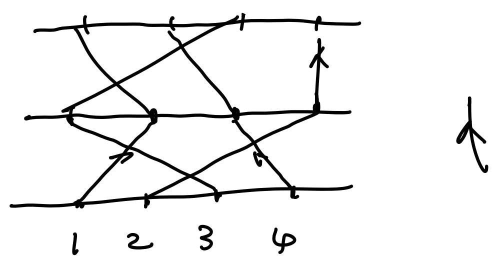
For a positive integer n, symmetric group on n elements S_n. which is the set of all permutations of the set $X = \{1, 2, \cdots, n\}$

$$|S_n| = n!$$

A permutation can be written as

$$\phi = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix} \quad \text{with } p_i = \phi(i)$$

$$\phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$



$$\phi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$\phi_2 \phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

right: $\phi_2 \cdot_R \phi_1 = \phi_2(\phi_1(x))$

$\phi_2 \cdot_L \phi_1 = \phi_1(\phi_2(x))$

$$\phi_2 \phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

Definition: $x \in X$. $\phi \in S_X$: ϕ fixes/moves x if

$\phi(x) = x \quad (\neq x)$

Definition. Let $i_1, \cdots i_r$ be distinct integers between 1 and n. If $\phi \in S_n$. s.t.

$\phi(i_1) = i_2, \quad \phi(i_2) = i_3 \quad \cdots \quad \phi(i_r) = i_1$

and fixes the rest.

We call ϕ r-cycle. $(i_1\, i_2\, i_3 \cdots i_r)$

A 2-cycle is called a transposition

$$\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix} = (1\,2\,3\,4) \qquad \text{4-cycle}$$

$$\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix} = (1)(2\,4\,3) = (2\,4\,3)$$

$$\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix} = (1\,2)(3\,4)$$

②	①	②①
$2 \to 1$	$1 \to 2$	$1 \to 1$
$4 \to 4$	$2 \to 4$	$2 \to 4$
$3 \to 2$	$4 \to 3$	$4 \to 2$
$1 \to 3$	$3 \to 1$	$3 \to 3$

$$\phi_2 \cdot \phi_1 = \begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix} = (1\,3\,2)(1\,2\,4\,3)$$

$$= (1)(2\,4)(3) = (2\,4)$$

Remarks:

1. $(234) = (423) = (342)$

2. disjoint cycles commute

$$(234)(56) = (56)(234)$$

3. inverse:

$$[(12)(345)]^{-1} = (345)^{-1}(12)^{-1}$$

$$= (543)(21)$$

$$= (12)(354)$$

Theorem: $\forall\, \phi \in S_n$ is either a cycle or can be factorized into disjoint cycles.

Proof (by induction): suppose true for $k\,(<n)$ moved points (starting from $k=0$, i.e. identity)

$$\phi:\quad x_1 \to x_2$$
$$x_2 \to x_3$$
$$\vdots$$
$$x_{j-1} \to x_j$$

r is the smallest number s.t.

$$x_{r+1} \in \{x_1, \ldots x_r\} \qquad (r \le n)$$

$x_{r+1} = \phi(x_r) = x_1$ otherwise for $i < r$,

$\phi(x_r) = x_i \quad r = i-1$ contradiction

⑤

Let $\sigma = (1, \cdots r)$ if $r = n$. $\phi = \sigma$; ϕ is a cycle

if $r < n$. $\phi = \sigma \cdot \phi'$ $\quad$ $\phi' Y = \phi Y$

ϕ' is a permutation than moves less than k points, which can be factorized by assumption

σ', $\sigma_1' \sigma_2'$ $\quad$ $Y = X \setminus \{x_1, \cdots x_r\}$

ϕ: factorized into cycles.

Remarks :

1. 1-cycles (1) usually suppressed

$(123)(4) = (123)$

2. (Def) a complete factorization includes one 1-cycle for each fixed x.

$(1)(234)$

3. A complete fact. is unique (except for the order of factors)

Why are we interested in S_n ?

Theorem (Cayley, 1878): Every group G is isomorphic to a subgroup of S_G ("can be imbeded" in S_G).

In particular, if $|G| = n$, then G is isomorphic to a subgroup of $\underline{S_n}$.

Note. $\underset{\uparrow}{S_G} \cong \underline{S_n} \leftarrow \{1, 2, 3 \cdot n\}$ ordered.

unordered elements

$G = \mu_N = \{1, \omega, \cdots \omega^{N-1}\}$ natural ordering

$\underline{D_n}$. $\underline{SU(n)}$ no natural ordering

Define a group action. $h \in G$:

$L(h) : G \to G$

$L(h) \cdot g \mapsto \underline{hg}$ $(\forall g \in G)$

$\boxed{L(h) \in S_G}$

$G = \{g_1, g_2, \cdots, g_n\}$

$hG = \{hg_1, hg_2, \cdots hg_n\}$

$\underline{hg_i \in G}$

$L(h_1) \cdot L(h_2) = L(h_1 h_2)$ homomorphism

$L : G \to \underline{\text{im}\, G} \subset S_G$

$h \to L(h)$

L is an $\underline{\text{isomorphism}}$ $(L(h_1) = L(h_2)$ iff $h_1 = h_2)$

Matrix rep. of S_n.

n-dim vectorspace. $\vec{e}_i = (0\,0\,0\,\underset{i\text{-th}}{1}\,\cdots\,0\,0)$

$$T(\phi) : \vec{e}_i \longrightarrow \vec{e}_{\phi(i)}$$

$$\sum x_i \vec{e}_i \to \sum x_i \vec{e}_{\phi(i)} = \sum x_{\phi^{-1}(i)} \vec{e}_i$$

$$T(\phi)\vec{e}_i = \sum_{j=1}^{n} A(\phi)_{ji} \vec{e}_j$$

Matrix rep. of a general finite group:

a finite group of order n is imbeded in $GL(n.K)$

Example.

1. $\mathbb{Z}_n \cong \langle (12, \cdots n) \rangle \xrightarrow{n=3} \{ 1, (123), (132) \}$ $\overset{A_3 \text{ defined later}}{\shortparallel}$

$$\mathbb{Z}_3 \cong A_3 < S_3$$

$$1 \to 1$$

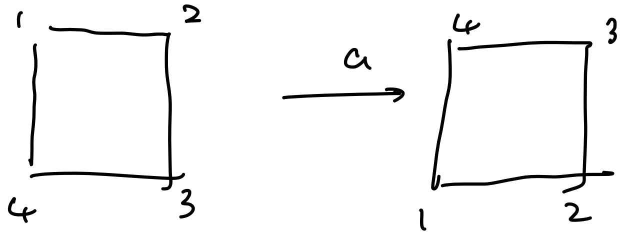
$\omega = e^{i\frac{2\pi}{3}}$ $\qquad \omega \to (123)$ $\qquad (123)(123) = (132)$

$$\omega^2 \to (132)$$

2. $D_4 := \langle a.b \mid a^4 = b^2 = (ab)^2 = 1 \rangle \qquad |D_4| = 8$

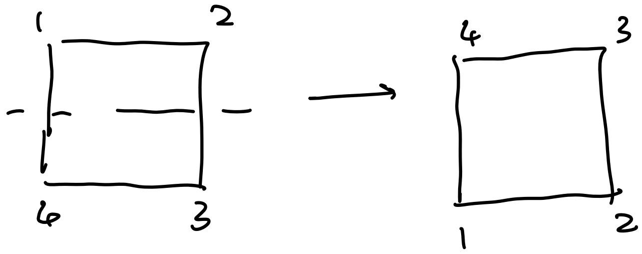
S_8

a.



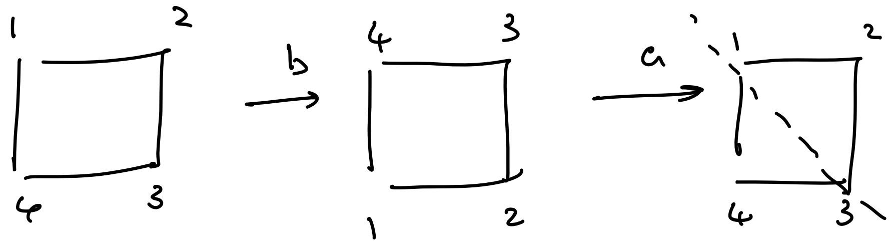
$:= (1234) \quad \times \underline{4}$

b



$:= (14)(23) \quad \times 2$

ab



$:= (24) \quad \times 2$

$D_4 \cong H \subset \underline{S_4} \subset \underline{S_8}$ — S_8 is the "upper limit"

can be imbeded in

$D_n \cong H \subset \underline{\underline{S_n}}$ — a much smaller S_n .

How to find the isomorphism ?

→ use <u>multiplication table</u> (Cayley table)

Klein 4-group $\quad V = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$

Symmetric ⟺ abelian.

		e	a	b	c(= ab)
1	e	e	a	b	c
2	a	a	e	c	b
3	b	b	c	e	a
4	c	c	b	a	e

$\phi : \quad V \to \mathrm{im}(V) \subset S_4$

$a \mapsto \phi(a)$

$e = (1, 1) \qquad c = (-1, -1)$

$a = (1, -1)$

$b = (-1, 1)$

mat. rep.

$\phi(e) = 1$

$\phi(a) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \underline{(12)(34)}$ $\quad a = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right)$

$\underline{\phi(b)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \underline{(13)(24)}$ $\quad b = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

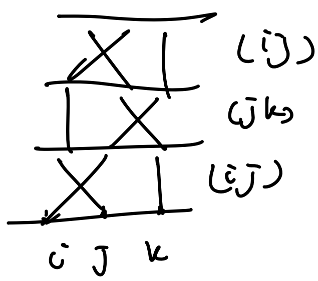
$\phi(c) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$

– $\underline{\text{Transpositions}}$. even & odd permutations

(2-cycle)

i, j, k are distinct

① $\underline{(ij)}\,\underline{(jk)}\,\underline{(ij)} = (ik) = (jk)(ij)(jk)$



$= (ik)$

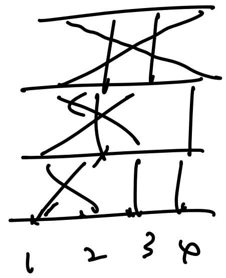
② $(ij)^2 = 1 \qquad (ij)^{-1} = (ij)$

③ $(ij)(kl) = (kl)\cdot(ij) \qquad \{i, j\} \cap \{k, l\} = \phi.$

$\underline{\text{Theorem}}$ Every permutation $\phi \in S_n$ is a product of transpositions.

Proof: $\phi \in S_n$ has a cycle decomposition.

each cycle. $(1,2,\cdots r) = (1\ r)(1, r-1)\cdots(1\ 2)$



$(14)(13)(12) = (1\,2\,3\,4)$

transpositions generate the symmetric group

Remarks.

1. Other ways to generate S_n:

① $\sigma_i := (i, i+1) \quad (1 \le i \le n-1)$

$$(i\,j) = \underbrace{(i\ i+1)}_{\sigma_i}(i+1 . j)\underbrace{(i\ i+1)}_{\sigma_i} \quad (i<j)$$

$|i-j| = k \qquad |i-j| = k-1$

by induction on k:

$\forall (i\,j) \Rightarrow \{\sigma_i\}$

② $(1\,2)$ & $(1\,2\cdots n)$

$(2\,3) = (1\,2\cdots n)(1\,2)(1\,2\cdots n)^{-1}$

$\overset{\text{generates}}{\Rightarrow} \{\sigma_i\} \underset{①}{\overset{\text{generates}}{\Rightarrow}}$ all cycles.

decomposition into transpositions are not unique:

$$(123)(4) = \underbrace{(13)(12)}_{2} = \underbrace{(23)(13)}_{2}$$

$$= \underbrace{(13)(42)(12)(14)}_{4}$$

$$= \underbrace{(13)(42)(12)(14)(23)(23)}_{6} \quad - -$$

number seems to preserve parity!

Definition A permutation $\phi \in S_n$ is even (odd) if it is a product of even (odd) number of transpositions

Definition: If $\phi = \sigma_1 \cdots \sigma_t \in S_n$ is a complete cycle decomposition.

$$\operatorname{sgn}(\phi) = (-1)^{n-t}$$

(complete fact. unique, well-defined func)

Remarks:

(1.) tansposition. $\tau = (ij)$: 1 2-cycle + $(n-2)$ 1-cycle

($(12) \in S_4$ $\quad (12)(3)(4) \quad t = 3$)

$$\operatorname{sgn}(\tau) = (-1)^{n-[(n-2)+1]} = -1$$

(2.) $\operatorname{sgn}(\tau\phi) = -\operatorname{sgn}(\phi)$ $\quad$ τ : trans.
ϕ: permutation

$\phi = \sigma_1 \sigma_2 \cdots \sigma_r$. $\tau = (ij)$

① $(ij)(i\,a_1 a_2 \cdots a_k\, j\, b_1 b_2 \cdots b_l)$

$= (i\,a_1 a_2 \cdots a_k)(j\,b_1 b_2 \cdots b_l)$

② $(ij)(i\,a_1 \cdots a_k)(j\,b_1 \cdots b_l) = (i\,a_1 a_2 \cdots a_k\, j\, b_1 \cdots b_l)$

(3) $sgn(\phi_1 \phi_2) = sgn(\phi_1)\cdot sgn(\phi_2)$

suppose holds for $m-1$: $\phi_1 = \tau_1 \cdots \tau_m$ minimal decomposition.

induction on m

$sgn(\phi_1 \phi_2) = sgn(\tau_1 \cdots \tau_m \phi_2) \overset{(2.)}{=} -sgn(\tau_2 \cdots \tau_m \phi_2)$

$= -sgn(\tau_2 \cdots \tau_m)\, sgn(\phi_2)$

$\overset{(2)}{=} sgn(\tau_1 \cdots \tau_m)\, sgn(\phi_2)$

$= sgn(\phi_1)\, sgn(\phi_2)$ $\rightarrow$ holds for m

4. construct homomorphism

$\epsilon : S_n \longrightarrow \mathbb{Z}_2$

$\phi \longmapsto sgn(\phi) = \pm 1$

[$\epsilon_{ijk} = sgn(ijk)$ in physics]

ϕ even $\rightarrow$ $sgn(\phi) = 1$

odd $\rightarrow$ $sgn(\phi) = -1$

Definition Alternating group $A_n \subset S_n$

is the subgroup of even permutations

~~odd~~ : (odd)(odd) = even. $\rightarrow$ not a subgroup

$A_2 = \{1\}$ $\cancel{(12)}$

$A_3 = \{\underline{1}, (123), (132)\}$

$A_4 = \{1, (123), (132)$
$(124)\ (142)$
$(134)\ (143)$
$(234)\ (243)$
$(12)(34), (13)(24)$
$(14)(23)\}$

$n=2.\ t=2$

$n=3.\ t=1$ or 3

1 3-cycle

3 1-cycles

$n=4.\ t=2$ or 4

1 4-cycle

1 1-cycle + 1 3-cycle

2 2-cycles