8. Unitarity and reducibility

We want to separate the representations constructed in Chapter 7 into smaller spaces preserved by symmetry. Unitary representations allow this through orthogonal complements, so we begin by constructing invariant inner products. The regular representation then provides a concrete setting in which to find invariant subspaces. Finally, we distinguish the irreducible types from the repeated copies of each type.

8.1. Unitarization and Haar measure

Reading. [Moore, §§11.4–11.5, pp. 231–240].

A representation is unitary for a chosen inner product when

$$\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle \qquad \forall g, v, w.$$

We seek an inner product with this property for a given representation.

Definition 8.1 (Unitarizable representation)**.** A complex representation (V, ρ) is *unitarizable* if there is a positive-definite Hermitian inner product on V for which it is unitary.

Permutation representations preserve the standard inner product, so they are already orthogonal over $\mathbb{R}$ and unitary over $\mathbb{C}$. The next theorem constructs an invariant inner product for every finite-dimensional representation of a finite group.

Theorem 8.2 (Unitarization for finite groups)**.** *Every finite-dimensional complex representation of a finite group is unitarizable.*

Proof. Start with any positive-definite inner product $\langle\ ,\ \rangle_0$ and put

$$\langle v, w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle_0. \tag{8.1}$$

Each summand is sesquilinear and Hermitian, so the average is too. For $v \neq 0$ all terms in $\langle v, v \rangle_G$ are nonnegative and the identity term is positive. Thus the average is positive definite. For $h \in G$,

$$\langle \rho(h)v, \rho(h)w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle \rho(gh)v, \rho(gh)w \rangle_0 = \langle v, w \rangle_G,$$

since $g \mapsto gh$ permutes the group. □

Remark 8.3 (Unitarization by a change of coordinates). Theorem 8.2 keeps the representation operators and changes the inner product. For a matrix representation, the same construction gives one change of coordinates that makes all its matrices unitary for the standard inner product. In a basis with $\langle v, w\rangle_0 = v^\dagger w$, define

$$Q = \frac{1}{|G|}\sum_{g\in G} D(g)^\dagger D(g).$$

Then $\langle v, w\rangle_G = v^\dagger Q w$ and

$$\begin{aligned} D(h)^\dagger Q D(h) &= \frac{1}{|G|}\sum_{g} (D(g)D(h))^\dagger D(g)D(h) \\ &= \frac{1}{|G|}\sum_{g} D(gh)^\dagger D(gh) = Q. \end{aligned}$$

The last equality reindexes the sum by $g \mapsto gh$. Also $Q^\dagger = Q$, and for $v \neq 0$,

$$v^\dagger Q v = \frac{1}{|G|}\sum_{g} \|D(g)v\|^2 \geq \frac{\|v\|^2}{|G|} > 0.$$

Diagonalize $Q = R\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)R^\dagger$ with R unitary and all $\lambda_i > 0$. Its positive square root is

$$S = Q^{1/2} = R\,\mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})R^\dagger, \qquad S^\dagger = S, \quad S^2 = Q.$$

The coordinates $\tilde{v} = Sv$ satisfy $\tilde{v}^\dagger \tilde{w} = v^\dagger S^\dagger S w = v^\dagger Q w = \langle v, w\rangle_G$. Transforming $D(g)v$ gives $SD(g)v = SD(g)S^{-1}\tilde{v}$, so the equivalent matrices in these coordinates are

$$\widetilde{D}(g) = SD(g)S^{-1}.$$

They are unitary because

$$\begin{aligned} \widetilde{D}(g)^\dagger \widetilde{D}(g) &= (S^{-1})^\dagger D(g)^\dagger S^\dagger S D(g) S^{-1} \\ &= S^{-1}D(g)^\dagger Q D(g) S^{-1} = S^{-1} Q S^{-1} = \mathbb{1}. \end{aligned}$$

The proof of Theorem 8.2 used an average that is unchanged when a group element multiplies its summation index. To extend that construction to continuous groups, we need an integral with the same property. For representations of continuous groups, we assume continuity; in finite dimensions this means continuity of each matrix entry, so the functions to be averaged are continuous.

For a finite group, left or right multiplication by a fixed h merely reorders the terms in the average,

$$\frac{1}{|G|}\sum_{g\in G} f(hg) = \frac{1}{|G|}\sum_{g\in G} f(g) = \frac{1}{|G|}\sum_{g\in G} f(gh).$$

For a continuous group, we replace the sum by an integral $\int_G f(g)\,d\mu(g)$. The problem is to choose the integration volume element $d\mu(g)$ so that the integral has the same invariance. We will then ask when it can be normalized to give an average.

Haar measure
Haar 测度

Definition 8.4 (Invariant integration and Haar measure)**.** For the finite groups and matrix Lie groups considered here, an integral with nonzero positive weights is *left invariant* if

$$\int_G f(hg)\,d\mu(g) = \int_G f(g)\,d\mu(g) \qquad \forall h \in G$$

for every nonnegative or integrable function f. The corresponding volume measure is called a *left Haar measure*. Right invariance replaces $f(hg)$ by $f(gh)$ and defines a *right Haar measure*.

To express this in terms of region volumes, let $A \subseteq G$ be a measurable subset, or region, of the group. Define its indicator function and its volume by

$$1_A(g) = \begin{cases} 1, & g \in A, \\ 0, & g \notin A, \end{cases} \qquad \mu(A) := \int_G 1_A(g)\,d\mu(g) = \int_A 1\,d\mu(g).$$

Thus $\mu(A)$ is a region's volume, while $d\mu(g)$ specifies the weights used when integrating a function.

The left translate $hA = \{ha : a \in A\}$ is another subset of G. Its indicator 1_{hA} is 1 on hA and 0 elsewhere. Since $g \in hA$ exactly when $h^{-1}g \in A$,

$$1_{hA}(g) = 1_A(h^{-1}g).$$

Apply left invariance to $f = 1_A$ and the element h^{-1},

$$\begin{aligned} \mu(hA) &= \int_G 1_{hA}(g)\,d\mu(g) \\ &= \int_G 1_A(h^{-1}g)\,d\mu(g) \\ &= \int_G 1_A(g)\,d\mu(g) = \mu(A). \end{aligned}$$

Thus invariance of the integral implies invariance of region volumes. Right invariance similarly gives $\mu(Ah) = \mu(A)$.

For a familiar example, take ordinary volume in $\mathbb{R}^3$ and let B_R be the solid ball of radius $R > 0$ centered at the origin. Denote a point by $P = (x, y, z)$. In spherical coordinates,

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

The Jacobian is $|\partial(x, y, z)/\partial(r, \theta, \phi)| = r^2\sin\theta$, so the same volume element has the two coordinate expressions

$$d\mu(P) = dV = dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$$

For an integrable function f, this means

$$\int_{B_R} f(P)\,dV = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{R} dr\, f(P(r, \theta, \phi))\, r^2 \sin\theta.$$

Setting $f = 1$ gives the volume of the ball,

$$\begin{aligned}\mu(B_R) &= \int_{B_R} 1 \, dV \\ &= \left(\int_0^R r^2 \, dr\right)\left(\int_0^\pi \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) \\ &= \frac{R^3}{3} \cdot 2 \cdot 2\pi = \frac{4\pi R^3}{3}.\end{aligned}$$

Here $r^2 \sin\theta$ is the weight that converts coordinate volume $dr\, d\theta\, d\phi$ into ordinary spatial volume.

For a matrix Lie group, a continuous positive weight similarly gives $d\mu(g) = w(q)\, dq_1 \cdots dq_d$ in local coordinates. The weight must account for how coordinate volume changes under group multiplication.

For a finite group, invariance also determines the weights in a sum $\sum_g w(g) f(g)$. Left invariance and the substitution $k = hg$ give

$$\sum_g w(g) f(g) = \sum_g w(g) f(hg) = \sum_k w(h^{-1}k) f(k).$$

Taking f to be one at a single element and zero elsewhere shows $w(k) = w(h^{-1}k)$. Hence all weights are the same. Normalizing their sum to one gives $w(g) = 1/|G|$ and $\mu(A) = |A|/|G|$.

For an infinite group, invariance alone does not guarantee that this normalization is possible. To form an average we also need finite, positive total volume,

$$0 < \mu(G) = \int_G 1 \, d\mu(g) < \infty.$$

Theorem 8.5 (Existence and normalization of Haar measure)**.** *Every finite group and every matrix Lie group has a left Haar measure, unique up to a positive constant. For a compact group it can be normalized by $\mu(G) = 1$. This normalized measure is also right invariant and invariant under inversion.*

Compactness is the condition that makes the invariant integral a finite average. Its meaning for matrix groups is concrete.

For a matrix group, *compact* has the familiar meaning *closed and bounded* in the finite-dimensional space of all matrix entries. A complex $n \times n$ matrix is a point of $\mathbb{R}^{2n^2}$, and this criterion is the Heine–Borel theorem. Closed means that the limit of a convergent sequence of group matrices still belongs to the group. Bounded means that one bound works for every entry of every group matrix. Equivalently, every sequence in a compact matrix group has a subsequence converging to an element of the group.

To show that $U(n)$ is closed, suppose a sequence $U_k \in U(n)$ converges entry by entry to an $n \times n$ matrix U. Taking the Hermitian conjugate and multiplying matrices commute with these limits, so

$$U^\dagger U = \left(\lim_{k\to\infty} U_k\right)^\dagger \left(\lim_{k\to\infty} U_k\right) = \lim_{k\to\infty} (U_k^\dagger U_k) = \mathbb{1}.$$

Thus the limit U is also unitary, which proves closedness. To show boundedness, use the unit-column condition,

$$\sum_i |U_{ij}|^2 = 1 \quad \Longrightarrow \quad |U_{ij}| \le 1 \qquad \text{for all } i, j,$$

so it is bounded. Thus $U(n)$ is compact, as are its closed subgroups $SU(n)$ and $SO(n)$.

Compact Lie groups	Noncompact Lie groups
$U(1) \cong SO(2)$	$(\mathbb{R}, +)$
$SO(3)$	$(\mathbb{R}_{>0}, \times)$
$O(n),\ SO(n)$	$GL(n, \mathbb{R})$
$U(n),\ SU(n)$	$SL(n, \mathbb{R}),\ n \ge 2$

The orthogonal and unitary groups are closed and bounded by the same argument as above. The translation and positive-scaling groups are unbounded. The two noncompact matrix groups contain, respectively,

$$t\mathbb{1}_n \in GL(n, \mathbb{R}), \qquad \operatorname{diag}(t, t^{-1}, 1, \dots, 1) \in SL(n, \mathbb{R}), \quad t > 0,$$

whose entries are unbounded as $t \to \infty$; the second family requires $n \ge 2$.

We use the existence and uniqueness theorem without proof. Haar measure is finite on compact regions. In particular, a compact group has finite, positive total volume $\mu(G)$, so dividing all volumes by $\mu(G)$ makes it one. *From now on, for compact groups, we use this normalized Haar measure and abbreviate $d\mu(g)$ by dg.* Thus

$$\int_G 1 \, dg = \mu(G) = 1.$$

To prove right invariance, fix h and define $\nu_h(A) = \mu(Ah)$, the volume of a region after right translation. Associativity and left invariance applied to the whole region Ah give

$$\nu_h(kA) = \mu(k(Ah)) = \mu(Ah) = \nu_h(A),$$
$$\nu_h(G) = \mu(Gh) = \mu(G) = 1.$$

Thus ν_h is another normalized left Haar measure and must equal μ, proving right invariance. To prove invariance under inversion, put $\nu(A) = \mu(A^{-1})$. Using $(kA)^{-1} = A^{-1}k^{-1}$ and the right invariance just proved,

$$\nu(kA) = \mu(A^{-1}k^{-1}) = \mu(A^{-1}) = \nu(A),$$
$$\nu(G) = \mu(G^{-1}) = \mu(G) = 1.$$

Uniqueness again gives $\nu = \mu$. Consequently

$$\int_G f(hg)\, dg = \int_G f(gh)\, dg = \int_G f(g^{-1})\, dg = \int_G f(g)\, dg.$$

For finite groups this convention is $\int_G f(g)\, dg = |G|^{-1} \sum_g f(g)$.

Example 8.6 (Elementary Haar measures). For addition on $\mathbb{R}$, translating x to $y = x + a$ gives $dy = dx$, so

$$\int_{\mathbb{R}} f(x+a)\,dx = \int_{\mathbb{R}} f(y)\,dy.$$

Thus $d\mu(x) = dx$ is a Haar measure. On $\mathbb{Z}$, the analogous integral is $\sum_{n\in\mathbb{Z}} f(n)$, invariant under reindexing $n \mapsto n + m$. This is counting measure, with weight one at each integer.

For multiplication on $\mathbb{R}_{>0}$, start with a density $w(x)\,dx$. The substitution $y = ax$, with $a > 0$, gives

$$\int_0^\infty f(ax)w(x)\,dx = \int_0^\infty f(y)\frac{w(y/a)}{a}\,dy.$$

Invariance for all f requires $w(y/a)/a = w(y)$. Taking $a = y$ gives $w(y) = w(1)/y$, so a Haar measure is $d\mu(x) = dx/x$. Indeed $d(ax)/(ax) = dx/x$.

For $U(1)$, write $g = e^{i\theta}$. Multiplication by $e^{i\phi}$ shifts θ to $\theta + \phi$ modulo 2π, so the invariant angular weight is constant, $d\mu = C\,d\theta$. Normalization fixes it,

$$1 = \mu(U(1)) = C\int_0^{2\pi} d\theta = 2\pi C, \qquad dg = d\mu(g) = \frac{d\theta}{2\pi}.$$

An arc of angular length $\Delta\theta$ therefore has measure $\Delta\theta/(2\pi)$, and

$$\int_{U(1)} f(g)\,dg = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta.$$

The first three groups are noncompact, and their Haar measures have infinite total measure.

Example 8.7 (Haar measure on $GL(n,\mathbb{R})$). Let $dA = \prod_{i,j} dA_{ij}$ be ordinary coordinate volume in the n^2 real matrix entries. Under left multiplication by a fixed $B \in GL(n,\mathbb{R})$,

$$A' = BA, \qquad A'_{ij} = \sum_k B_{ik}A_{kj}, \qquad \frac{\partial A'_{ij}}{\partial A_{k\ell}} = B_{ik}\delta_{j\ell}.$$

Ordering the coordinates column by column makes the Jacobian matrix $\operatorname{diag}(B,\dots,B)$, with n blocks. Hence $dA' = |\det B|^n dA$. The density $|\det A|^{-n}$ cancels this factor,

$$\frac{dA'}{|\det A'|^n} = \frac{|\det B|^n dA}{|\det B \det A|^n} = \frac{dA}{|\det A|^n}.$$

Thus $d\mu(A) = dA/|\det A|^n$ is left invariant. For $A' = AB$, each row transforms by B, again giving $dA' = |\det B|^n dA$. The same determinant cancellation proves right invariance.

Example 8.8 (Different left and right measures). The affine transformations of the line are

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \qquad x > 0, \quad y \in \mathbb{R},$$

with multiplication $(x,y)(x',y') = (xx', y + xy')$. Left multiplication by (a,b) sends (x,y) to $(ax, b+ay)$ and has Jacobian determinant a^2. Thus

$$\frac{dx'\,dy'}{(x')^2} = \frac{a^2\,dx\,dy}{(ax)^2} = \frac{dx\,dy}{x^2}.$$

Right multiplication sends (x,y) to $(ax, y+bx)$. Its Jacobian is

$$\det\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} = a, \qquad \frac{dx'\,dy'}{x'} = \frac{a\,dx\,dy}{ax} = \frac{dx\,dy}{x}.$$

Hence $d\mu_L = dx\,dy/x^2$ and $d\mu_R = dx\,dy/x$. Their ratio depends on x, so they are not constant multiples. Left and right invariance need not coincide for a noncompact group.

We now have the continuous analogue of the normalized sum used in Theorem 8.2. Applying it to the inner-product function $g \mapsto \langle \rho(g)v, \rho(g)w\rangle_0$ gives the same unitarization construction.

Theorem 8.9 (Compact groups)**.** *Every continuous finite-dimensional complex representation of a compact group is unitarizable.*

Proof. Starting with any positive-definite Hermitian inner product, define

$$\langle v, w\rangle_G = \int_G \langle \rho(g)v, \rho(g)w\rangle_0 \, dg.$$

For fixed v, w, the integrand is continuous on compact G, hence bounded. Since $\int_G dg = 1$, this integral is finite. For $v \neq 0$, the continuous function $g \mapsto \|\rho(g)v\|_0^2$ is strictly positive on the compact group and has a positive minimum. The integral is therefore positive. Sesquilinearity and Hermitian symmetry pass through the integral. For $h \in G$, right invariance gives

$$\langle \rho(h)v, \rho(h)w\rangle_G = \int_G \langle \rho(gh)v, \rho(gh)w\rangle_0 \, dg = \langle v, w\rangle_G.$$

□

Example 8.10 (A representation that cannot be made unitary). Recall the one-dimensional representation $\rho(A) = \det A$ of $GL(n,\mathbb{R})$ from §7.3, acting on $\mathbb{C}$. Every positive-definite inner product on this space has the form $\langle z, w\rangle = c\bar{z}w$, with $c > 0$. Thus

$$\langle \rho(A)z, \rho(A)z\rangle = |\det A|^2 \langle z, z\rangle.$$

For $A = 2\mathbb{1}_n$, the factor is 2^{2n}, whatever c is. This representation is not unitarizable. The compactness assumption in Theorem 8.9 matters: the Haar measure of $GL(n,\mathbb{R})$ has infinite total volume, so the normalized averaging argument does not apply.

To evaluate invariant averages for rotational symmetries, we next need an explicit coordinate formula for Haar measure on $SU(2)$.

Example 8.11 (Haar measure on $SU(2)$). Use the z-y-z Euler parametrization from (3.1), written explicitly as

$$\begin{aligned} u(\alpha,\beta,\gamma) &= e^{-\mathrm{i}\alpha\sigma^3/2}e^{-\mathrm{i}\beta\sigma^2/2}e^{-\mathrm{i}\gamma\sigma^3/2} \\ &= \begin{pmatrix} e^{-\mathrm{i}(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-\mathrm{i}(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{\mathrm{i}(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{\mathrm{i}(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}. \end{aligned}$$

Take $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, and $\gamma \in [0, 4\pi)$. For $0 < \beta < \pi$, the two nonzero complex entries determine $(\alpha+\gamma)/2$ and $(\alpha-\gamma)/2$ modulo 2π. Consequently the coordinate identifications are

$$(\alpha,\gamma) \sim (\alpha+4\pi,\gamma) \sim (\alpha,\gamma+4\pi) \sim (\alpha+2\pi,\gamma+2\pi).$$

The chosen rectangle is a fundamental domain for this lattice: reduce α modulo 2π using the simultaneous shift, then reduce γ modulo 4π. It counts each generic group element once. At $\beta = 0, \pi$, one complex entry vanishes and its phase is undetermined.

The first row of u is (a, b), where

$$a = e^{-\mathrm{i}(\alpha+\gamma)/2}\cos\frac{\beta}{2}, \qquad b = -e^{-\mathrm{i}(\alpha-\gamma)/2}\sin\frac{\beta}{2}.$$

The condition $|a|^2 + |b|^2 = 1$ identifies $SU(2)$ with the unit sphere $S^3 \subset \mathbb{C}^2$. Differentiate the amplitudes and phases,

$$\begin{aligned} da &= e^{-\mathrm{i}(\alpha+\gamma)/2}\left[-\tfrac{1}{2}\sin\tfrac{\beta}{2}\,d\beta - \tfrac{\mathrm{i}}{2}\cos\tfrac{\beta}{2}(d\alpha+d\gamma)\right], \\ db &= -e^{-\mathrm{i}(\alpha-\gamma)/2}\left[\tfrac{1}{2}\cos\tfrac{\beta}{2}\,d\beta - \tfrac{\mathrm{i}}{2}\sin\tfrac{\beta}{2}(d\alpha-d\gamma)\right]. \end{aligned}$$

In each absolute square, the cross term between the real amplitude change and imaginary phase change vanishes. Hence

$$\begin{aligned} ds^2 &= |da|^2 + |db|^2 = \tfrac{1}{2}\operatorname{Tr}(du^\dagger du) \\ &= \tfrac{1}{4}\left[d\beta^2 + \cos^2\frac{\beta}{2}(d\alpha+d\gamma)^2 + \sin^2\frac{\beta}{2}(d\alpha-d\gamma)^2\right] \\ &= \tfrac{1}{4}\big(d\beta^2 + d\alpha^2 + d\gamma^2 + 2\cos\beta\,d\alpha\,d\gamma\big). \end{aligned}$$

Multiplication of u on either side by a fixed unitary matrix preserves the trace expression, so it preserves this length element and its volume. If $ds^2 = \sum_{ij} M_{ij}\,dq_i\,dq_j$, the coordinate volume element is $\sqrt{\det M}\,dq_1\,dq_2\,dq_3$. Here $q = (\alpha,\beta,\gamma)$ and

$$M = \frac{1}{4}\begin{pmatrix} 1 & 0 & \cos\beta \\ 0 & 1 & 0 \\ \cos\beta & 0 & 1 \end{pmatrix}, \qquad \det M = \frac{1-\cos^2\beta}{4^3} = \frac{\sin^2\beta}{64}.$$

Thus the invariant volume element is

$$d\mathrm{vol}_{S^3} = \frac{\sin\beta}{8}\,d\alpha\,d\beta\,d\gamma.$$

Its integral over the above domain is $\frac{1}{8}(2\pi)\int_0^\pi \sin\beta\,d\beta\,(4\pi) = 2\pi^2$. Dividing by this volume gives the normalized Haar measure,

$$du = \frac{\sin\beta}{16\pi^2}\,d\alpha\,d\beta\,d\gamma. \tag{8.2}$$

The coordinate degeneracies at $\beta = 0, \pi$ have measure zero. Moore uses a different Euler convention, with σ^1 in the middle factor; throughout these notes we use the above z-y-z convention.

The quotient spaces $SO(3) \cong SU(2)/\{\mathbb{1}, -\mathbb{1}\}$ and $S^2 \cong SO(3)/SO(2)$ describe rotations and directions. To use the measure just computed for averages on these spaces, we need to pass from integration on G to integration on G/H.

For compact G and a closed subgroup H, a function F on G/H lifts to the right-H-invariant function $g \mapsto F(gH)$ on G. To assign a volume to a region B of cosets, take the volume of all group elements whose cosets lie in B. With $p(g) = gH$, this means $\mu_{G/H}(B) = \mu_G(p^{-1}(B))$ and gives

$$\int_{G/H} F(gH)\,d(gH) = \int_G F(gH)\,dg.$$

Here $d(gH)$ abbreviates $d\mu_{G/H}$. Use normalized Haar measure on H as well. For a continuous function f on G, averaging its values within each coset defines

$$(P_H f)(gH) = \int_H f(gh)\,dh.$$

Integrating these coset averages recovers the original integral,

$$\int_G f(g)\,dg = \int_{G/H}\left(\int_H f(gh)\,dh\right) d(gH). \tag{8.3}$$

To see that the inner integral depends only on gH, replace g by gh_0 and use left invariance on H,

$$\int_H f(gh_0h)\,dh = \int_H f(gh)\,dh.$$

Its integral over G/H is therefore the integral of the same function over G. Interchanging the two integrals and using right invariance gives

$$\int_G\int_H f(gh)\,dh\,dg = \int_H\left(\int_G f(gh)\,dg\right)dh = \int_H\left(\int_G f(g)\,dg\right)dh = \int_G f(g)\,dg.$$

Example 8.12 (From $SU(2)$ to rotations and directions). Under $SU(2) \to SO(3)$ the two elements u and $-u$ give the same rotation. Averaging over the subgroup $\{\mathbb{1}, -\mathbb{1}\}$ gives

$$(P_H f)(uH) = \tfrac{1}{2}\big(f(u) + f(-u)\big).$$

In Euler coordinates, increasing γ by 2π changes u to $-u$. A function lifted from $SO(3)$ therefore has period 2π in γ. Its integral over $0 \le \gamma < 4\pi$ is twice the integral over $0 \le \gamma < 2\pi$, so $2/(16\pi^2) = 1/(8\pi^2)$ gives

$$dR = \frac{\sin\beta}{8\pi^2}\,d\alpha\,d\beta\,d\gamma, \qquad 0 \le \alpha, \gamma < 2\pi, \quad 0 \le \beta \le \pi. \tag{8.4}$$

For a function of the direction Re_3, the final rotation about e_3 has no effect, so the angle γ drops out. Its integral contributes $\int_0^{2\pi} d\gamma = 2\pi$, changing the prefactor to $2\pi/(8\pi^2) = 1/(4\pi)$. Thus the normalized sphere integral is

$$\int_{SO(3)} F(Re_3)\, dR = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} F(\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta) \sin\beta\, d\beta\, d\alpha.$$

This is integration on the coset space $SO(3)/SO(2) \cong S^2$ from §4.2.

8.2. The regular representation

Reading. [Moore, §11.6, pp. 241–243].

We now construct a representation directly from G itself, as we did for finite groups in Definition 7.22. Its vectors are functions on G, and group multiplication translates their arguments. To use the inner-product methods just developed, we require these functions to have finite squared norm under Haar integration. This gives the space $L^2(G)$.

Definition 8.13 (The Hilbert space $L^2(G)$)**.** For a compact group with normalized Haar measure, $L^2(G)$ consists of square-integrable complex functions, meaning

$$\|f\|_2^2 = \int_G |f(x)|^2\, dx < \infty, \qquad \langle f_1, f_2 \rangle = \int_G \overline{f_1(x)} f_2(x)\, dx.$$

Here dx denotes the same normalized Haar measure, with x as the integration variable. Functions that differ only on a set of Haar measure zero represent the same element; this is what *equal almost everywhere* means. With this identification $L^2(G)$ is a Hilbert space.

For a finite group, $L^2(G)$ is the $|G|$-dimensional space of function values. The delta functions $\delta_h(x) = \delta_{hx}$ form an orthogonal basis, with $\|\delta_h\|_2^2 = 1/|G|$; $\sqrt{|G|}\,\delta_h$ is the corresponding orthonormal basis. Thus, when the complex basis $\mathbf{e}_h$ in Definition 7.22 is taken orthonormal, the map $\mathbf{e}_h \mapsto \sqrt{|G|}\,\delta_h$ is a unitary identification with $L^2(G)$.

Definition 8.14 (Left and right regular representations)**.** For $a, b \in G$ and $f \in L^2(G)$, define

$$(L(a)f)(x) = f(a^{-1}x), \qquad (R(b)f)(x) = f(xb).$$

These are the *left* and *right regular representations*.

Their composition laws are

$$(L(a)L(a')f)(x) = f(a'^{-1}a^{-1}x) = (L(aa')f)(x),$$
$$(R(b)R(b')f)(x) = f(xbb') = (R(bb')f)(x).$$

Haar invariance gives

$$\|L(a)f\|_2^2 = \int_G |f(a^{-1}x)|^2\,dx = \|f\|_2^2,$$
$$\|R(b)f\|_2^2 = \int_G |f(xb)|^2\,dx = \|f\|_2^2.$$

The first equality of integrals uses $y = a^{-1}x$ and left invariance; the second uses $y = xb$ and right invariance. In each case $dx = dy$ denotes equality of the corresponding Haar integration elements. Their inverses are $L(a^{-1})$ and $R(b^{-1})$, so both are unitary. They also commute,

$$(L(a)R(b)f)(x) = f(a^{-1}xb) = (R(b)L(a)f)(x).$$

They therefore give a representation of $G \times G$,

$$((L(a)R(b))f)(x) = f(a^{-1}xb). \tag{8.5}$$

For finite G, the two actions on the delta-function basis are

$$L(a)\delta_h = \delta_{ah}, \qquad R(b)\delta_h = \delta_{hb^{-1}}.$$

Both L and R are representations acting on vectors in the function space from the left; their names refer to the side on which the argument x is translated.

We must also check that the regular action varies continuously with the group element. Functions in $L^2(G)$ need not be continuous at each point, so we measure the change using their norm.

An L^2 limit means convergence in mean square, $\int_G |f_n - f|^2\,dx \to 0$. It need not give convergence at every point. A subspace is *dense* if every vector can be approximated arbitrarily closely in norm by vectors of that subspace. Continuous functions are dense in $L^2(G)$.

For a Hilbert-space representation, continuity means *strong continuity*, $\|\rho(g)v - \rho(g_0)v\| \to 0$ as $g \to g_0$, for every fixed v. For a continuous function F, compactness gives uniform convergence $L(a)F \to F$ as $a \to e$, hence convergence in L^2. For general $f \in L^2(G)$, choose a continuous F close in norm. Unitarity and the triangle inequality give

$$\|L(a)f - f\|_2 \le 2\|f - F\|_2 + \|L(a)F - F\|_2.$$

Both terms can be made arbitrarily small. Continuity at any a_0 follows by translating this estimate, and the same argument applies to R.

Example 8.15 (Left and right translations on C_3). Let $C_3 = \{e, a, a^2\}$, where $a^3 = e$, and use the ordered basis $(\delta_e, \delta_a, \delta_{a^2})$. Left multiplication cycles these basis functions in the order

$$\delta_e \xrightarrow{L(a)} \delta_a \xrightarrow{L(a)} \delta_{a^2} \xrightarrow{L(a)} \delta_e.$$

Right translation instead gives $R(a)\delta_e = \delta_{a^{-1}} = \delta_{a^2}$. Thus

$$L(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad R(a) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = L(a)^{-1}.$$

For instance, $L(a)(f(e), f(a), f(a^2))^T = (f(a^2), f(e), f(a))^T$, exactly as prescribed by $f(a^{-1}x)$. The inverse in the function argument produces the forward shift of the delta-function label.

Example 8.16 (The regular representation of $\mathbb{Z}_2$). For $G = \{e, s\}$, write a function as $(f(e), f(s))^T$. Then

$$L(s) = R(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The functions $(1, 1)^T$ and $(1, -1)^T$ have norm one in the normalized L^2 inner product and transform with eigenvalues $+1$ and -1.

The two lines in the $\mathbb{Z}_2$ example each carry a representation. To find such pieces for other groups, we look for subspaces preserved by translations.

irreducible representation
不可约表示

Definition 8.17 (Invariant subspace and irreducibility)**.** A subspace $W \subset V$ is *invariant* if $\rho(g)W \subset W$ for every $g \in G$. Since $\rho(g^{-1})$ also preserves W, we have $\rho(g)W = W$. Restricting each $\rho(g)$ to W therefore gives a representation on W. A nonzero representation is *irreducible* if its only invariant subspaces are 0 and V. It is *reducible* if it has a nonzero proper invariant subspace. For unitary Hilbert-space representations, invariant subspaces in this definition are required to be closed.

The group may move vectors within an invariant subspace. On the fixed-vector space V^G, every group element acts as the identity. For example, the line spanned by $(1, -1)^T$ above is invariant, although s reverses the sign of each of its vectors.

One way to construct an invariant subspace is to start with a vector and span all its images under the group. This also gives a useful test for irreducibility.

Definition 8.18 (Cyclic vector)**.** A vector v is *cyclic* if $\operatorname{span}\{\rho(g)v : g \in G\} = V$. For Hilbert spaces the closure of this span is used.

The span of an orbit is invariant, since

$$\rho(h)\Big(\sum_i c_i \rho(g_i)v\Big) = \sum_i c_i \rho(hg_i)v \in \operatorname{span}\{\rho(g)v : g \in G\}.$$

Thus every nonzero vector of an irreducible representation is cyclic. Conversely, if every nonzero vector is cyclic, an invariant subspace containing such a vector must contain its orbit span and hence all of V. In the Hilbert-space case, take closures and require the invariant subspace to be closed.

Example 8.19 (A cyclic vector in a reducible representation). Let S_n, with $n \geq 2$, permute the basis of $\mathbb{C}^n$. The orbit of e_1 contains every e_i, so e_1 is cyclic. Nevertheless, the line spanned by $e_1 + \cdots + e_n$ is invariant, because a permutation only reorders the sum. The representation is therefore reducible, although e_1 is cyclic.

To relate the regular representation to a given representation, regard each of its matrix entries as a function on G. If translations preserve the span of these functions, they supply a finite-dimensional piece of the regular representation. We can check this directly from the representation law.

Let (V, ρ) be a finite-dimensional unitary representation, with matrices $D(x)$ in an orthonormal basis. The entries $x \mapsto D_{ij}(x)$, called *matrix coefficients*, provide finite-dimensional invariant subspaces of $L^2(G)$. They are continuous, hence square-integrable on compact G. The representation law gives

$$(L(a)D_{ij})(x) = D_{ij}(a^{-1}x) = \sum_k D_{ik}(a^{-1})D_{kj}(x),$$
$$(R(b)D_{ij})(x) = D_{ij}(xb) = \sum_k D_{ik}(x)D_{kj}(b).$$

For a fixed row index i, put

$$\mathscr{R}_i = \operatorname{span}\{D_{i1}, \dots, D_{id}\}, \qquad d = \dim V.$$

The second formula expresses $R(b)D_{ij}$ as a linear combination of the functions D_{ik} with numerical coefficients $D_{kj}(b)$. Thus $\mathscr{R}_i$ is invariant under the right regular action. Similarly, for a fixed column index j,

$$\mathscr{L}_j = \operatorname{span}\{D_{1j}, \dots, D_{dj}\}$$

is invariant under the left regular action by the first formula. Allowing both indices to vary gives a finite-dimensional subspace invariant under $G \times G$. These functions can be linearly dependent. For the two-dimensional trivial representation $D(x) = \mathbb{1}_2$, all entries are constant or zero, and their span is only one-dimensional.

The coefficient functions above depend on the chosen basis. To express their relation to the regular action without choosing a basis, use the operator space $\operatorname{End}(V)$. It carries the $G \times G$ action

$$(a, b) \cdot A = \rho(a) A \rho(b)^{-1}.$$

Indeed,

$$\begin{aligned}(a,b)\cdot\big((a',b')\cdot A\big) &= \rho(a)\rho(a')A\rho(b')^{-1}\rho(b)^{-1} \\ &= \rho(aa')A\rho(bb')^{-1} = ((aa', bb')\cdot A).\end{aligned}$$

We seek a linear map from these operators to functions on G that respects the two group actions. The trace gives $\iota_V : \operatorname{End}(V) \to L^2(G)$ by

$$\iota_V(A)(x) = \operatorname{Tr}(A\rho(x^{-1})). \tag{8.6}$$

If E_{ij} sends the jth basis vector to the ith, then $(E_{ij})_{ab} = \delta_{ai}\delta_{bj}$. Expanding the trace gives

$$\begin{aligned}\iota_V(E_{ij})(x) &= \sum_{a,b} (E_{ij})_{ab} D_{ba}(x^{-1}) \\ &= \sum_{a,b} \delta_{ai}\delta_{bj} D_{ba}(x^{-1}) = D_{ji}(x^{-1}).\end{aligned}$$

Since $D^\vee(x) = D(x^{-1})^{\mathsf{T}}$, these are coefficients of the dual representation $V^\vee$; replacing V by $V^\vee$ recovers the functions $D_{ij}(x)$. For example, if $G = U(1)$ and $\rho(z) = z$, then $\iota_V(1)(z) = z^{-1}$, so the dual label matters. The inverse in $\rho(x^{-1})$ makes the map compatible with our left/right translation convention, as the following calculation shows,

$$\begin{aligned}(L(a)R(b)\iota_V(A))(x) &= \operatorname{Tr}\big(A\rho((a^{-1}xb)^{-1})\big)\\ &= \operatorname{Tr}(A\rho(b)^{-1}\rho(x^{-1})\rho(a))\\ &= \operatorname{Tr}(\rho(a)A\rho(b)^{-1}\rho(x^{-1})).\end{aligned}$$

The last step uses cyclicity of the trace. Hence ι_V intertwines (8.5) with $A \mapsto \rho(a)A\rho(b)^{-1}$ on $\operatorname{End}(V)$. Chapter 9 proves that for irreducible V these maps identify the matrix-coefficient subspaces of $L^2(G)$.

8.3. Reducible and irreducible representations

Reading. [Moore, §11.7, pp. 243–251].

An invariant subspace supplies one smaller representation. To separate its action from the remaining coordinates, we must determine what invariance implies for the full representation matrices. Let W be an invariant subspace of a finite-dimensional representation V. Choose a basis of W and extend it to a basis of V. The matrices have the form

$$D(g) = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix}. \tag{8.7}$$

Indeed, a vector with coordinates $(w, 0)$ must stay in W, so its lower coordinates remain zero. The block $A(g)$ represents the action on W. For a general vector,

$$D(g)\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} A(g)w + B(g)u \\ C(g)u \end{pmatrix}.$$

Multiplying two group matrices gives

$$D(g)D(h) = \begin{pmatrix} A(g)A(h) & A(g)B(h) + B(g)C(h) \\ 0 & C(g)C(h) \end{pmatrix} = D(gh).$$

Comparing diagonal blocks shows $A(gh) = A(g)A(h)$ and $C(gh) = C(g)C(h)$. The lower coordinates ignore additions of vectors from W. The quotient space V/W describes precisely this remaining information, and $C(g)$ gives its group action.

Definition 8.20 (Quotient representation)**.** For an invariant subspace W, the quotient space V/W carries the representation

$$\rho_{V/W}(g)(v + W) = \rho(g)v + W.$$

If $v' = v + w$ with $w \in W$, then $\rho(g)v' - \rho(g)v = \rho(g)w \in W$, so the formula is independent of the coset representative. The action laws follow from those on V,

$$\rho_{V/W}(g)\rho_{V/W}(h)(v + W) = \rho(gh)v + W.$$

In the quotient basis induced by the remaining basis vectors, its matrices are $C(g)$ in (8.7). Those remaining vectors need not span an invariant subspace of V; the block $B(g)$ records their components along W.

To obtain independent blocks, we need a complement U that is also invariant. Then $V = W \oplus U$, and bases chosen within the two summands give $\operatorname{diag}(A(g), C(g))$. If this splitting can be continued until every summand is irreducible, the entire representation is built from independent irreducible pieces.

complete reducibility
完全可约性

Definition 8.21 (Complete reducibility)**.** A finite-dimensional representation is *completely reducible* if it is a direct sum of irreducible representations. It is *indecomposable* if it cannot be written as the direct sum of two nonzero invariant subspaces.

Irreducibility implies indecomposability; the converse can fail.

Remark 8.22 (Invariant complements in matrix form). The off-diagonal blocks give a test for whether an invariant complement exists. A complement to W projects bijectively onto the lower coordinate space. Thus each lower coordinate vector u has exactly one representative in the complement, whose upper coordinates depend linearly on u. It has the form $U_X = \{(Xu, u) : u \in \kappa^{\dim V - \dim W}\}$ for a linear map X. Applying the group action gives

$$D(g)\begin{pmatrix} Xu \\ u \end{pmatrix} = \begin{pmatrix} A(g)Xu + B(g)u \\ C(g)u \end{pmatrix}.$$

This lies in U_X for every u precisely when $A(g)X + B(g) = XC(g)$. The corresponding basis change is

$$S = \begin{pmatrix} \mathbb{1} & X \\ 0 & \mathbb{1} \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} \mathbb{1} & -X \\ 0 & \mathbb{1} \end{pmatrix}.$$

Multiplying first on the right and then on the left gives

$$D(g)S = \begin{pmatrix} A(g) & A(g)X + B(g) \\ 0 & C(g) \end{pmatrix},$$
$$S^{-1}D(g)S = \begin{pmatrix} A(g) & A(g)X + B(g) - XC(g) \\ 0 & C(g) \end{pmatrix}.$$

Thus one fixed X must satisfy $B(g) = XC(g) - A(g)X$ for every g. An invariant complement removes all off-diagonal blocks by the same change of basis.

For a unitary representation, an invariant complement can be found without solving the matrix condition for X. Orthogonality supplies the candidate,

$$W^{\perp} = \{u \in V : \langle u, w \rangle = 0 \text{ for every } w \in W\}.$$

Unitarity ensures that the group action preserves this subspace.

Proposition 8.23. *For a unitary representation, the orthogonal complement of an invariant subspace is invariant. If W is closed in a Hilbert space, then $V = W \oplus W^{\perp}$.*

Proof. For $u \in W^{\perp}$ and $w \in W$,

$$\langle \rho(g)u, w\rangle = \langle u, \rho(g)^{\dagger} w\rangle = \langle u, \rho(g^{-1})w\rangle = 0,$$

since $\rho(g^{-1})w \in W$. Thus $\rho(g)u \in W^{\perp}$. The orthogonal decomposition holds for every closed subspace of a Hilbert space, and in particular for every finite-dimensional subspace. □

Theorem 8.24 (Complete reducibility)**.** *Every finite-dimensional unitary representation is completely reducible. Consequently every continuous finite-dimensional complex representation of a compact group, including a finite group, is completely reducible.*

Proof. If V is reducible, split it as $W \oplus W^{\perp}$ with both summands invariant. Repeat on reducible summands. Their dimensions decrease, so the process terminates with irreducible summands. For compact groups first apply Theorem 8.9. □

Example 8.25 (Translations on $L^2(\mathbb{R})$). The proof ends because the dimensions of the summands decrease. There is no such termination argument for an infinite-dimensional Hilbert space. For example, translations on $L^2(\mathbb{R})$ act by

$$(L(t)f)(x) = f(x - t)$$

and are unitary for Lebesgue measure. Their Fourier description uses a continuous parameter $k \in \mathbb{R}$, rather than a discrete sum of one-dimensional subspaces. Indeed, with the Fourier convention $\widehat{f}(k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ikx} f(x)\,dx$, first defined for integrable square-integrable functions and then extended to L^2 by continuity,

$$\begin{aligned}\widehat{L(t)f}(k) &= (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ikx} f(x - t)\,dx \\ &= e^{-ikt}(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iky} f(y)\,dy = e^{-ikt}\widehat{f}(k).\end{aligned}$$

The formal translation eigenfunctions e^{ikx} are not in $L^2(\mathbb{R})$: their absolute value is one on a set of infinite measure. In fact, there is no nonzero one-dimensional invariant subspace. If f spanned such a line, $L(1)f = \lambda f$ for some $|\lambda| = 1$, so $(e^{-ik} - \lambda)\widehat{f}(k) = 0$ almost everywhere. The set where $e^{-ik} = \lambda$ is countable and has measure zero. Thus $\widehat{f} = 0$ in L^2, contradicting $f \neq 0$. Compact groups have a different outcome: Chapter 9 proves that their strongly continuous unitary Hilbert-space representations are Hilbert direct sums of finite-dimensional irreducibles.

Having established decomposition, we can identify the irreducible pieces in the simplest case: an abelian group. Its representation matrices commute, allowing us to find invariant lines by simultaneous diagonalization.

Proposition 8.26 (Abelian groups)**.** *A finite-dimensional unitary representation of an abelian group over* $\mathbb{C}$ *is a direct sum of one-dimensional representations.*

Proof. The unitary matrices $D(g)$ form a commuting family of normal matrices. An eigenspace of one member is invariant under every other member, since $D(g)v = \lambda v$ implies

$$D(g)D(h)v = D(h)D(g)v = \lambda D(h)v.$$

If every member is scalar, any orthonormal basis gives invariant lines. Otherwise diagonalize a nonscalar member and restrict the others to its eigenspaces. Repeat within any eigenspace on which some member is nonscalar. Dimensions decrease, so eventually every member is scalar on each remaining subspace. An orthonormal basis of those subspaces gives invariant lines. □

Example 8.27 (The characters of $U(1)$)*.* The one-dimensional continuous representations are

$$\rho_n(e^{\mathrm{i}\theta}) = e^{\mathrm{i}n\theta}, \qquad n \in \mathbb{Z}.$$

In one dimension, unitarizability forces $|\rho(z)| = 1$. Lifting along $\theta \mapsto e^{\mathrm{i}\theta}$ therefore gives a continuous homomorphism $\mathbb{R} \to U(1)$, which has the form $\theta \mapsto e^{\mathrm{i}k\theta}$. One obtains this by choosing a continuous argument near zero; the homomorphism law makes that argument locally additive, hence linear, and then determines it on all of $\mathbb{R}$. For it to descend to $U(1)$, periodicity requires $e^{2\pi \mathrm{i}k} = 1$, so $k \in \mathbb{Z}$. Unitarization and Proposition 8.26 show that every continuous finite-dimensional complex representation of $U(1)$ is a direct sum of these characters.

Example 8.28 (Real and complex irreducibility)*.* The complex-field assumption in the preceding proposition matters. The defining representation of $SO(2)$ on $\mathbb{R}^2$ is

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

It has no invariant real line, since a rotation through $\pi/2$ takes any real line to its perpendicular line. It is therefore irreducible over $\mathbb{R}$. Over $\mathbb{C}$, the vectors $(1, -\mathrm{i})^T$ and $(1, \mathrm{i})^T$ span invariant lines with eigenvalues $e^{\mathrm{i}\theta}$ and $e^{-\mathrm{i}\theta}$. Irreducibility depends on the field.

Permutation representations provide higher-dimensional irreducible pieces as well. Let S_n permute the basis vectors of κ^n, $\rho(\sigma)e_i = e_{\sigma(i)}$, with $\kappa = \mathbb{R}$ or $\mathbb{C}$. The invariant vector $u_0 = \sum_i e_i$ gives the decomposition

$$\kappa^n = \kappa u_0 \oplus V_{\mathrm{std}}, \qquad V_{\mathrm{std}} = \{(x_1, \dots, x_n) : \sum_i x_i = 0\}.$$

The first summand is trivial and the second is the *standard representation.* Explicitly, every $x \in \kappa^n$ splits as

$$x = \frac{\sum_i x_i}{n} u_0 + \left(x - \frac{\sum_i x_i}{n} u_0 \right).$$

The second term has coordinate sum zero. The two subspaces intersect only at zero, and permutations preserve each one.

Proposition 8.29. *For $n \geq 2$, the standard representation of S_n is irreducible over $\mathbb{R}$ and $\mathbb{C}$.*

Proof. Let $U \subset V_{\mathrm{std}}$ be nonzero and invariant, and take $x \in U$, $x \neq 0$. Some coordinates x_i, x_j differ, since equal coordinates and $\sum_i x_i = 0$ would imply $x = 0$. Hence

$$x - \rho((i\,j))x = (x_i - x_j)(e_i - e_j) \in U.$$

Since $x_i - x_j \neq 0$, division gives $e_i - e_j \in U$. Permuting indices then gives every $e_a - e_b \in U$. In particular, $e_1 - e_n, \dots, e_{n-1} - e_n$ form a basis of V_{std}, so $U = V_{\mathrm{std}}$. □

Example 8.30 (Matrices for the standard representation of S_3). We can now write the standard representation explicitly and see how a change of coordinates makes its matrices unitary, as in Remark 8.3. Put $s = (1\,2)$ and $t = (2\,3)$. For $u_1 = e_1 - e_2$, $u_2 = e_2 - e_3$, permuting coordinates gives

$$su_1 = -u_1, \quad su_2 = u_1 + u_2, \qquad tu_1 = u_1 + u_2, \quad tu_2 = -u_2.$$

These images are the columns of the matrices,

$$D(s) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad D(t) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

The representation is unitary for the inherited inner product, with $\langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle = 2$ and $\langle u_1, u_2 \rangle = -1$. Its Gram matrix is therefore $Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The displayed matrices need not themselves be unitary in a nonorthonormal basis.

Gram–Schmidt gives a vector orthogonal to u_1,

$$w_2 = u_2 - \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} u_1 = u_2 + \tfrac{1}{2} u_1, \qquad \|w_2\|^2 = 2 + \tfrac{1}{2} - 1 = \tfrac{3}{2}.$$

Normalizing u_1 and w_2 gives the orthonormal basis

$$v_1 = \frac{e_1 - e_2}{\sqrt{2}}, \qquad v_2 = \frac{e_1 + e_2 - 2e_3}{\sqrt{6}}.$$

Since $v_1 = u_1/\sqrt{2}$ and $v_2 = (u_1 + 2u_2)/\sqrt{6}$, the basis-change matrix and the transformed matrices are

$$S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{pmatrix}, \qquad D_2(g) = S^{-1} D(g) S.$$

Here the columns of S express the new basis in the old one, so vector coordinates satisfy $c_{\mathrm{old}} = S c_{\mathrm{new}}$. The identity $D(g) S c_{\mathrm{new}} = S D_2(g) c_{\mathrm{new}}$ then gives the displayed similarity transformation. Thus

$$D_2(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D_2(t) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{8.8}$$

Both are real orthogonal matrices, and the three-cycle $r = st = (1\,2\,3)$ is represented by

$$D_2(r) = D_2(s) D_2(t) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = R(2\pi/3).$$

Thus $D_2(s)^2 = D_2(t)^2 = (D_2(s) D_2(t))^3 = \mathbb{1}_2$.

The unitarity assumption made orthogonal complements invariant. The following examples show how a preserved subspace can fail to have any invariant complement when that assumption is unavailable.

Example 8.31 (Shears). The additive group $\mathbb{R}$ has the representation on κ^2, with $\kappa = \mathbb{R}$ or $\mathbb{C}$,

$$\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad \rho(x)\rho(y) = \rho(x+y).$$

The line $W = \kappa e_1$ is invariant. Every line other than W is spanned by some $(a, 1)^T$. Its image is $(a+x, 1)^T$, which belongs to that line only if $x = 0$. Thus W is the only invariant line, and it has no invariant complement. The representation is reducible and indecomposable. In the matrix criterion above, $A = C = 1$ and $B = x$, so the required identity $B = XC - AX$ would read $x = X - X = 0$ for every x. On W and on V/W the action is trivial, but the action on V is not a direct sum of these two trivial representations.

Example 8.32 (A shear representation from the determinant). We can transfer the shear example to a matrix group through a homomorphism. For $A \in GL(n, \mathbb{R})$, the additive homomorphism $\ell(A) = \log|\det A|$ gives another such representation,

$$\rho(A) = \begin{pmatrix} 1 & \ell(A) \\ 0 & 1 \end{pmatrix}.$$

The homomorphism law is explicit,

$$\rho(A)\rho(B) = \begin{pmatrix} 1 & \ell(A) + \ell(B) \\ 0 & 1 \end{pmatrix} = \rho(AB).$$

Since $\ell(\operatorname{diag}(e^x, 1, \dots, 1)) = x$, the image contains every shear from the preceding example and has the same absence of an invariant complement.

Example 8.33 (Euclidean transformations in homogeneous coordinates). Euclidean motions give a spatial example of the same obstruction. Adjoining one coordinate turns the affine action $x \mapsto Rx + t$ into a linear action of $SE(3)$ on $\mathbb{R}^4$,

$$\rho(R, t) = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}, \qquad \rho(R, t)\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Rx + t \\ 1 \end{pmatrix}.$$

The subspace $W = \{(x, 0) : x \in \mathbb{R}^3\}$ is invariant. Rotations act on it by their defining representation, and translations act trivially. The one-dimensional quotient is trivial. A complement would be spanned by some $(a, 1)$. Translations send it to $(a + t, 1)$, so no such line is invariant. This representation is not completely reducible.

8.4. Isotypic components and multiplicities

A completely reducible representation can contain several equivalent copies of an irreducible representation. The copies have the same symmetry action after

suitable identifications, while occupying independent subspaces. We therefore distinguish the irreducible type from the label of its copy.

Suppose V is a finite-dimensional, completely reducible complex representation. Group its irreducible summands by equivalence class, choose one representative (V_μ, ρ_μ) for each class, and let m_μ count its copies. Then

$$V \cong \bigoplus_{\mu} \left(\bigoplus_{a=1}^{m_\mu} V_\mu \right).$$

Only finitely many labels occur, since V is finite-dimensional.

isotypic component
等型分量

Definition 8.34 (Multiplicity and isotypic component)**.** The integer m_μ is the *multiplicity* of V_μ in V. The sum of all irreducible subspaces equivalent to V_μ is its *isotypic component* $V[\mu]$.

To separate the copy label from the vector within that copy, introduce $\mathbb{C}^{m_\mu}$ with standard basis $e_1, \dots, e_{m_\mu}$. The tensor $e_a \otimes v$ represents v in the ath copy, through the linear map

$$\mathbb{C}^{m_\mu} \otimes V_\mu \longrightarrow \bigoplus_{a=1}^{m_\mu} V_\mu, \qquad e_a \otimes v \longmapsto (0, \dots, v, \dots, 0),$$

where v occupies the ath slot. The inverse sends $(v_1, \dots, v_{m_\mu})$ to $\sum_a e_a \otimes v_a$. Acting on the ath copy sends v to $\rho_\mu(g)v$, so under this identification the action is $\mathbb{1}_{m_\mu} \otimes \rho_\mu(g)$. Combining all irreducible types gives

$$\begin{aligned} V &\cong \bigoplus_{\mu} (\mathbb{C}^{m_\mu} \otimes V_\mu), \\ \rho(g) &\cong \bigoplus_{\mu} (\mathbb{1}_{m_\mu} \otimes \rho_\mu(g)). \end{aligned} \tag{8.9}$$

The factor $\mathbb{C}^{m_\mu}$ is called a *multiplicity space*. The group acts trivially on this factor because it leaves the copy label unchanged. Counting the basis vectors gives $\dim V = \sum_\mu m_\mu d_\mu$, with $d_\mu = \dim V_\mu$.

Within $V[\mu]$, use basis vectors $e_a \otimes v_i$, where $a = 1, \dots, m_\mu$ labels the copy and $i = 1, \dots, d_\mu$ labels a basis of V_μ. Then

$$\rho(g)(e_a \otimes v_j) = \sum_i D^\mu_{ij}(g) e_a \otimes v_i.$$

The index a labels the copy and stays unchanged under G. Changing the basis of $\mathbb{C}^{m_\mu}$ mixes equivalent copies. Schur's lemmas in Chapter 9 will give its intrinsic description as $\mathrm{Hom}_G(V_\mu, V)$ and show that $V[\mu]$ does not depend on the chosen irreducible decomposition.

Example 8.35 (Multiplicities in the regular representation of S_3). The trivial, sign, and standard two-dimensional representations are the three irreducible representations of S_3 over $\mathbb{C}$; their completeness is proved in Example 9.26. Their

characters on e, a transposition s, and a three-cycle r are respectively $(1,1,1)$, $(1,-1,1)$, and $(2,0,-1)$. The last triple follows by taking traces of the matrices above.

The regular representation has dimension six and character zero away from e. If its multiplicities are m_1, m_ϵ, m_2, addition of characters gives

$$\begin{aligned} m_1 + m_\epsilon + 2m_2 &= 6, \quad g = e, \\ m_1 - m_\epsilon &= 0, \quad g = s, \\ m_1 + m_\epsilon - m_2 &= 0, \quad g = r. \end{aligned}$$

The second equation gives $m_1 = m_\epsilon$ and the third gives $m_2 = 2m_1$. Substitution into the first gives $6m_1 = 6$, so

$$L^2(S_3) \cong V_1 \oplus V_\epsilon \oplus (\mathbb{C}^2 \otimes V_2).$$

The standard irrep occurs twice, accounting for four of the six dimensions. The factor $\mathbb{C}^2$ labels these two copies.

Example 8.36 (Two-site translation sectors). A ring with two sites per unit cell has two basis vectors at each momentum, $|k_m, A\rangle$ and $|k_m, B\rangle$. Both transform under a translation by ℓ cells as $e^{-\mathrm{i}k_m\ell}$. Thus the momentum sector contains two copies of the same one-dimensional representation,

$$V_m \cong \mathbb{C}^2 \otimes V_{k_m}, \qquad T^\ell|_{V_m} = e^{-\mathrm{i}k_m\ell}\mathbb{1}_2.$$

Translation leaves the two-dimensional multiplicity factor unchanged. A Hamiltonian can mix $|k_m, A\rangle$ and $|k_m, B\rangle$ while preserving translation symmetry, because every 2×2 matrix commutes with the scalar translation matrices on this sector.

Key points

- Averaging an inner product over a finite or compact group makes a finite-dimensional representation unitary.
- An invariant subspace gives an upper-triangular block form. An invariant complement gives a direct sum and block-diagonal form.
- In a unitary representation, an invariant closed subspace has an invariant orthogonal complement. In finite dimension this proves complete reducibility.
- Left and right translations commute on $L^2(G)$. Matrix coefficients span finite-dimensional invariant subspaces.
- In $V[\mu] \cong \mathbb{C}^{m_\mu} \otimes V_\mu$, the group acts on the irreducible factor and leaves the multiplicity factor fixed.

Key terms. unitary representation 幺正表示 · Haar measure Haar 测度 · regular representation 正则表示 · irreducible representation 不可约表示 · complete reducibility 完全可约性 · isotypic component 等型分量 · multiplicity 重数