

Part II.

Representation theory

7. Representations

Part I described symmetry operations and their group laws. We now study their action on vector spaces. The vectors may be quantum-state amplitudes or components of a tensor. A representation assigns a linear operator to each group element so that operator composition follows the group multiplication.

The choice of vector space matters. A rotation acts on a position vector, on a wavefunction, and on a spin state through different representations. We begin with symmetry operators and Wigner's theorem, then define representations and their matrix forms. Translation symmetry on a finite ring gives a worked example of how a symmetry-adapted basis simplifies a Hamiltonian.

7.1. SYMMETRY OPERATORS AND PHYSICAL SYSTEMS

READING. [Moore, §11.1, pp. 218–224]; [Weinberg, §2.2].

A quantum state vector is a nonzero vector $\psi \in \mathcal{H}$, where $\mathcal{H}$ is a complex Hilbert space. We use ϕ for another vector in the same space. Our matrix examples use $\mathcal{H} = \mathbb{C}^n$.

inner product
内积

Definition 7.1 (Inner product and Hilbert space). A complex *inner product* on a complex vector space $\mathcal{H}$ is a map

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$$

sesquilinear
半双线性
conjugate-linear
共轭线性

with the following properties. It is *sesquilinear*, meaning *conjugate-linear* in its first argument and linear in its second. Explicitly, for all $a, b \in \mathbb{C}$ and vectors $\phi, \phi_1, \phi_2, \psi, \psi_1, \psi_2 \in \mathcal{H}$,

$$\begin{aligned} \langle a\phi_1 + b\phi_2, \psi \rangle &= \bar{a} \langle \phi_1, \psi \rangle + \bar{b} \langle \phi_2, \psi \rangle, \\ \langle \phi, a\psi_1 + b\psi_2 \rangle &= a \langle \phi, \psi_1 \rangle + b \langle \phi, \psi_2 \rangle. \end{aligned}$$

conjugate symmetry
共轭对称性
positive definiteness
正定性

Here the bar denotes complex conjugation. The inner product also satisfies *conjugate symmetry* and *positive definiteness*,

$$\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}, \quad \langle \psi, \psi \rangle \geq 0, \quad \langle \psi, \psi \rangle = 0 \iff \psi = 0.$$

norm
范数
normalized
归一化

It defines the *norm* $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$. A vector is *normalized* when $\|\psi\| = 1$. A *Hilbert space* is an inner-product space *complete* in this norm. This means that every *Cauchy sequence* (ψ_m) , satisfying $\|\psi_m - \psi_n\| \rightarrow 0$ as $m, n \rightarrow \infty$, converges in norm to a vector of $\mathcal{H}$.

Hilbert space
希尔伯特空间
complete
完备
Cauchy sequence
柯西序列

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For column vectors $\phi = (\phi_1, \dots, \phi_n)^T$ and $\psi = (\psi_1, \dots, \psi_n)^T$ in $\mathbb{C}^n$, the standard inner product is

$$\langle \phi, \psi \rangle = \sum_{j=1}^n \overline{\phi_j} \psi_j.$$

adjoint
伴随算符

For a linear operator $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$, its *adjoint* $A^\dagger$ is the unique linear operator satisfying

$$\langle A\phi, \psi \rangle = \langle \phi, A^\dagger \psi \rangle \quad \forall \phi, \psi \in \mathbb{C}^n.$$

In an orthonormal basis, $A^\dagger = \overline{A}^T$ is the conjugate transpose of the matrix of A .

ray
射线

Definition 7.2 (Ray and transition probability). Let $\psi \in \mathcal{H}$ be a nonzero vector. Its *ray* is the one-dimensional complex subspace

$$[\psi] = \{z\psi : z \in \mathbb{C}\}.$$

transition probability
跃迁概率

A pure quantum state is specified by a ray, represented by any of its nonzero vectors. For nonzero vectors $\phi, \psi \in \mathcal{H}$, the *transition probability* between their rays is

$$p([\phi], [\psi]) = \frac{|\langle \phi, \psi \rangle|^2}{\|\phi\|^2 \|\psi\|^2}.$$

For normalized representatives this is $|\langle \phi, \psi \rangle|^2$.

Multiplying either representative by a nonzero scalar leaves this probability unchanged. Normalized vectors ψ and $e^{i\theta}\psi$ therefore describe the same state. A quantum symmetry is a bijection of the rays that preserves these transition probabilities.

unitary operator
么正算符

Definition 7.3 (Unitary operator). A *unitary* operator $U : \mathcal{H} \rightarrow \mathcal{H}$ is a bijective complex-linear map satisfying

$$\langle U\phi, U\psi \rangle = \langle \phi, \psi \rangle.$$

In an orthonormal basis, $U^\dagger U = \mathbb{1}$ and $U^{-1} = U^\dagger$.

A unitary operator preserves norms and transition probabilities. For a Hermitian Hamiltonian H on a finite-dimensional state space, a unitary operator U is a *symmetry of H* if it preserves all Hamiltonian matrix elements,

$$\langle U\phi, HU\psi \rangle = \langle \phi, H\psi \rangle \quad \forall \phi, \psi \in \mathcal{H}.$$

For $\phi = \psi$ normalized, this says that the mean energy is unchanged. Using the adjoint and $U^\dagger = U^{-1}$, the left-hand side is

$$\langle U\phi, HU\psi \rangle = \langle \phi, U^{-1}HU\psi \rangle.$$

Equality for all ϕ, ψ gives $U^{-1}HU = H$, or equivalently

$$UHU^{-1} = H, \quad HU = UH.$$

If $H\psi = E\psi$, then $H(U\psi) = U(H\psi) = E(U\psi)$, so U maps each energy eigenspace to itself. Conversely, if $U\psi = \lambda\psi$, then $U(H\psi) = H(U\psi) = \lambda H\psi$. Thus H preserves every eigenspace of U . A basis grouped by these symmetry eigenvalues makes H block diagonal.

Example 7.4 (A symmetric double well). Consider a particle in a symmetric double-well potential, $V(-x) = V(x)$. In a two-state approximation, take orthonormal states $|L\rangle$ and $|R\rangle$ localized in the left and right wells. A state in this subspace has the form

$$|\psi\rangle = c_L|L\rangle + c_R|R\rangle, \quad \psi = \begin{pmatrix} c_L \\ c_R \end{pmatrix}.$$

Choose the phases of the basis states so that reflection about the midpoint exchanges them,

$$S|L\rangle = |R\rangle, \quad S|R\rangle = |L\rangle, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^2 = \mathbb{1}_2.$$

The matrix S is unitary, and the identity and reflection form $\mathbb{Z}_2$. With equal well energies ϵ and tunnelling amplitude $t > 0$, the Hamiltonian is

$$H = \begin{pmatrix} \epsilon & -t \\ -t & \epsilon \end{pmatrix} = \epsilon \mathbb{1}_2 - tS.$$

Hence $HS = SH$. The even and odd combinations

$$|+\rangle = \frac{|L\rangle + |R\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|L\rangle - |R\rangle}{\sqrt{2}}$$

satisfy $S|\pm\rangle = \pm|\pm\rangle$, so

$$H|+\rangle = (\epsilon - t)|+\rangle, \quad H|-\rangle = (\epsilon + t)|-\rangle.$$

Reflection separates the two-dimensional state space into even and odd one-dimensional subspaces. The symmetry fixes these combinations; ϵ and t determine their energies. On the two subspaces, the same reflection acts by the scalars $+1$ and -1 , respectively.

The assignment $e \mapsto \mathbb{1}_2$, $s \mapsto S$ preserves multiplication in $\mathbb{Z}_2$, since $s^2 = e$ and $S^2 = \mathbb{1}_2$. This is a *representation* on $\text{span}\{|L\rangle, |R\rangle\}$, as we will define formally in §7.2.

For a spinless wavefunction in the position representation, time reversal acts by complex conjugation. In particular, it reverses a plane-wave factor,

$$\overline{e^{ikx}} = e^{-ikx}.$$

This introduces a second kind of symmetry operator.

Definition 7.5 (Antiunitary operator). An *antiunitary* operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is a bijective conjugate-linear map,

$$A(a\phi + b\psi) = \bar{a}A\phi + \bar{b}A\psi, \quad a, b \in \mathbb{C},$$

satisfying

$$\langle A\phi, A\psi \rangle = \overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle.$$

antiunitary operator
反么正算符

7. Representations

Choose an orthonormal basis of $\mathbb{C}^n$ and define componentwise conjugation,

$$K\psi = \bar{\psi}, \quad K^2 = \mathbb{1}.$$

The operator K is antiunitary. Every antiunitary operator A on $\mathbb{C}^n$ can be written as

$$A = UK, \quad (7.1)$$

where U is unitary. Indeed, set $U = AK$. For $a, b \in \mathbb{C}$ and $\phi, \psi \in \mathbb{C}^n$,

$$\begin{aligned} U(a\phi + b\psi) &= A(\bar{a}K\phi + \bar{b}K\psi) \\ &= aU\phi + bU\psi. \end{aligned}$$

Thus U is linear. It also preserves the inner product,

$$\langle U\phi, U\psi \rangle = \overline{\langle K\phi, K\psi \rangle} = \langle \phi, \psi \rangle.$$

Since A and K are bijective, U is unitary, and $K^2 = \mathbb{1}$ gives $UK = AK^2 = A$.

For a matrix M ,

$$KMK = \bar{M},$$

where the bar means conjugating each matrix entry. Consequently

$$A^2 = UKUK = U\bar{U}. \quad (7.2)$$

The bar here is different from the adjoint $U^\dagger$.

An antiunitary operator A is a symmetry of H when

$$AHA^{-1} = H, \quad \text{equivalently} \quad HA = AH.$$

Its action then conjugates the Hamiltonian matrix elements,

$$\langle A\phi, HA\psi \rangle = \langle A\phi, AH\psi \rangle = \overline{\langle \phi, H\psi \rangle}.$$

Energy expectation values are real, so they are unchanged.

Example 7.6 (Antiunitary squares and Kramers degeneracy). Conjugation itself has $K^2 = \mathbb{1}$. For a spin- $\frac{1}{2}$ state in the σ_3 eigenbasis, the spin operators are $S_j = \frac{\hbar}{2}\sigma_j$. Time reversal reverses angular momentum, so its operator must satisfy

$$TS_jT^{-1} = -S_j \quad (j = 1, 2, 3).$$

Componentwise conjugation changes the sign of only the second Pauli matrix,

$$\bar{\sigma}_1 = \sigma_1, \quad \bar{\sigma}_2 = -\sigma_2, \quad \bar{\sigma}_3 = \sigma_3.$$

Set

$$J = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = JK.$$

Since $J^\dagger J = \mathbb{1}_2$, the operator T is antiunitary. Conjugation by J reverses σ_1 and σ_3 and leaves σ_2 unchanged. Combining it with K therefore gives

$$T\sigma_jT^{-1} = J\bar{\sigma}_jJ^{-1} = -\sigma_j \quad (j = 1, 2, 3).$$

Thus $T = i\sigma_2 K$ has the required spin-reversal action. Since J is real and $J^2 = -\mathbb{1}_2$,

$$T^2 = J\bar{J} = -\mathbb{1}_2.$$

More generally, let T be an antiunitary symmetry of a Hermitian Hamiltonian H , with $TH = HT$ and $T^2 = -\mathbb{1}$. If $H\psi = E\psi$ with $\psi \neq 0$, then

$$H(T\psi) = T(H\psi) = T(E\psi) = E(T\psi),$$

since E is real. Thus $T\psi$ has the same energy as ψ . Antiunitarity gives

$$\langle T\psi, T^2\psi \rangle = \langle T\psi, \psi \rangle.$$

The left side is also $-\langle T\psi, \psi \rangle$, so $\langle T\psi, \psi \rangle = 0$. Since T preserves norms, $T\psi$ is nonzero. Every energy eigenstate therefore has an orthogonal partner of the same energy. This is *Kramers degeneracy*.

Both kinds of operator preserve norms and the absolute square of the inner product. They therefore act on rays and preserve transition probabilities. Wigner's theorem gives the converse.

Kramers degeneracy
克拉默斯简并

Theorem 7.7 (Wigner). *Let $\mathcal{H}$ be a nonzero complex Hilbert space. Every bijection s of its rays that preserves transition probabilities is implemented by a unitary or an antiunitary operator A ,*

$$s([\psi]) = [A\psi] \quad \forall \psi \neq 0.$$

Wigner's theorem
维格纳定理

If two operators A and B implement the same ray symmetry s and are both unitary or both antiunitary, then $B = e^{i\theta} A$ for some $\theta \in \mathbb{R}$.

For proofs, see Weinberg [Weinberg, §2.2, pp. 52–53, and Chapter 2, Appendix A, pp. 91–96] or the six-step proof of Simon et al. [SMCS, §III].

When $\mathcal{H} = \mathbb{C}$, there is only one ray. For every $\theta \in \mathbb{R}$, both the unitary map $z \mapsto e^{i\theta} z$ and the antiunitary map $z \mapsto e^{i\theta} \bar{z}$ induce the identity on this ray. For $\dim \mathcal{H} \geq 2$, a fixed ray symmetry has implementations of only one type, unitary or antiunitary. Complex conjugation can now change relative phases between components and hence change the ray. For example, in $\mathbb{C}^2$,

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \bar{\psi} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

These vectors are not proportional, so the identity and complex conjugation induce different transformations of rays.

Unitary symmetry operators form a subgroup and act by complex-linear maps. We first develop the theory of linear representations. For an antiunitary operator $A = UK$, conjugation remains part of composition,

$$(U_1 K)(U_2 K) = U_1 \bar{U}_2.$$

Passing to the unitary factors alone therefore need not preserve the group law. For a symmetry group containing both kinds of operator, we study the linear action of its unitary subgroup together with the additional relations imposed by the antiunitary symmetries.

7. Representations

7.2. REPRESENTATIONS AND EQUIVALENCE

READING. [Moore, §11.2,
pp. 225–229]; [Fulton, §1.1].

linear representation
线性表示

The essential definitions are restated here so that Part II can be read independently. Let κ denote $\mathbb{R}$ or $\mathbb{C}$. Unless specified otherwise, the representation spaces are finite-dimensional. Linear representations of topological groups are taken to be continuous.

Definition 7.8 (Linear representation). Let G be a group and V a vector space over κ . Recall from §3.5 that $GL(V)$ is the group of invertible linear maps $V \rightarrow V$, with composition as multiplication. A *representation* of G on V is a homomorphism

$$\rho : G \longrightarrow GL(V).$$

As in Definition 3.1, this means

$$\rho(g_1 g_2) = \rho(g_1) \rho(g_2) \quad \forall g_1, g_2 \in G. \quad (7.3)$$

representation space / carrier space
表示空间／载体空间

The space V is the *representation space* or *carrier space*, and $\dim V$ is the *dimension* of the representation. We write (V, ρ) for the representation.

The homomorphism property gives

$$\rho(e) = \mathbb{1}_V, \quad \rho(g^{-1}) = \rho(g)^{-1}.$$

Products act *right to left*, so $\rho(g_1 g_2)v$ means first applying $\rho(g_2)$ and then $\rho(g_1)$.

Equivalently, this is the group action of Definition 4.3 with $X = V$ and with the additional requirement of linearity,

$$\begin{aligned} g \cdot v &:= \rho(g)v, \\ e \cdot v &= v, \quad g_1 \cdot (g_2 \cdot v) = (g_1 g_2) \cdot v, \\ g \cdot (av + bw) &= a(g \cdot v) + b(g \cdot w). \end{aligned}$$

These conditions also ensure that every g acts invertibly, since its inverse is the action of g^{-1} .

Distinct group elements can act by the same operator.

trivial representation
平凡表示

Example 7.9 (Trivial representation). For any group G and vector space V , the *trivial representation* is

$$\rho_{\text{triv}}(g) = \mathbb{1}_V \quad \forall g \in G.$$

Every vector is fixed, and $\ker \rho_{\text{triv}} = G$. In one dimension, $\rho_{\text{triv}}(g) = 1$.

faithful representation
忠实表示

Definition 7.10 (Faithful representation). A representation is *faithful* if ρ is injective, or equivalently if

$$\ker \rho = \{g \in G : \rho(g) = \mathbb{1}_V\} = \{e\}.$$

This is faithfulness of the linear action in Definition 4.11.

The first isomorphism theorem gives $G/\ker \rho \cong \rho(G)$. In particular, a faithful representation makes G isomorphic to its image $\rho(G)$ and distinguishes every group element.

Definition 7.11 (Unitary representation). A representation (V, ρ) on a complex inner-product space is *unitary* if every $\rho(g)$ is a unitary operator,

$$\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle \quad \forall g \in G, \quad \forall v, w \in V.$$

Equivalently, $\rho(g)^\dagger = \rho(g^{-1})$.

For real inner-product spaces, the corresponding term is *orthogonal representation*.

Definition 7.12 (Matrix representation). Let $b = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ be an ordered basis of V . The *representation matrix* $D_b(g)$ is defined by

$$\rho(g)\mathbf{e}_j = \sum_{i=1}^n D_b(g)_{ij} \mathbf{e}_i.$$

Thus column j contains the coordinates of $\rho(g)\mathbf{e}_j$. The map $g \mapsto D_b(g)$ is the matrix representation in this basis.

Write a general vector as $v = \sum_j v_j \mathbf{e}_j$, with coordinate column $[v]_b = (v_1, \dots, v_n)^T$. Linearity gives

$$\rho(g)v = \sum_j v_j \rho(g)\mathbf{e}_j = \sum_i \left(\sum_j D_b(g)_{ij} v_j \right) \mathbf{e}_i.$$

Thus the coordinates transform by the usual matrix multiplication,

$$(\rho(g)v)_i = \sum_j D_b(g)_{ij} v_j, \quad [\rho(g)v]_b = D_b(g)[v]_b.$$

The sum contracts the second matrix index with the vector components. This is an *active* transformation of v in the fixed basis b .

Applying g_2 and then g_1 to a basis vector gives

$$\rho(g_1)\rho(g_2)\mathbf{e}_j = \sum_{i,k} D_b(g_1)_{ik} D_b(g_2)_{kj} \mathbf{e}_i.$$

Comparing coefficients with $\rho(g_1 g_2)\mathbf{e}_j$ yields

$$D_b(g_1 g_2) = D_b(g_1) D_b(g_2), \quad D_b(e) = \mathbb{1}_n.$$

For a unitary representation and an orthonormal basis b , these matrices also satisfy $D_b(g)^\dagger D_b(g) = \mathbb{1}_n$.

A *passive* change of basis keeps v fixed. Write the new basis as

$$\mathbf{e}'_j = \sum_i S_{ij} \mathbf{e}_i, \quad S \in GL(n, \kappa).$$

unitary representation
幺正表示

orthogonal representation
正交表示

matrix representation
矩阵表示

representation matrix
表示矩阵

active
主动

passive
被动

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The columns of S are the new basis vectors in the old coordinates. The two coordinate columns therefore satisfy

$$[v]_b = S[v]_{b'}, \quad [v]_{b'} = S^{-1}[v]_b.$$

If $\mathbf{e}'_j = \rho(g_0)\mathbf{e}_j$ for a fixed g_0 , then $S = D_b(g_0)$, so the passive transformation of coordinates uses $D_b(g_0)^{-1}$, as for rotated axes. Applying $\rho(g)$ to a general vector and expressing it in both bases gives

$$S[\rho(g)v]_{b'} = D_b(g)S[v]_{b'},$$

so

$$D_{b'}(g) = S^{-1}D_b(g)S \quad \forall g \in G. \quad (7.4)$$

The same matrix S changes the basis for every group element.

The linear maps $V_1 \rightarrow V_2$ form a vector space $\text{Hom}_K(V_1, V_2)$, abbreviated to $\text{Hom}(V_1, V_2)$ when the field is understood. A linear map $V \rightarrow V$ is an *endomorphism*; the space of these maps is $\text{End}(V) = \text{Hom}(V, V)$. Its invertible elements form $GL(V)$.

endomorphism
自同态

intertwiner
交织算符

Definition 7.13 (Intertwiner and equivalence). Let (V_1, ρ_1) and (V_2, ρ_2) be representations of the same group G . A linear map $A : V_1 \rightarrow V_2$ is an *intertwiner* if

$$A\rho_1(g) = \rho_2(g)A \quad \forall g \in G. \quad (7.5)$$

The representations are *equivalent* if such an A is invertible. In that case

$$\rho_2(g) = A\rho_1(g)A^{-1} \quad \forall g \in G.$$

equivalent
等价

The intertwining condition says that the two orders of operation agree,

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ \rho_1(g) \downarrow & & \downarrow \rho_2(g) \\ V_1 & \xrightarrow{A} & V_2. \end{array}$$

An invertible intertwiner identifies the two vector spaces while respecting every symmetry operation. Equation (7.4) is an instance of equivalence, with the coordinate map $A = S^{-1}$.

The intertwiners form a subspace $\text{Hom}_G(V_1, V_2) \subset \text{Hom}(V_1, V_2)$. For one representation,

$$\text{End}_G(V) = \text{Hom}_G(V, V) = \{A \in \text{End}(V) : A\rho(g) = \rho(g)A \quad \forall g \in G\}.$$

An invariant Hamiltonian is therefore an element of $\text{End}_G(V)$.

Example 7.14 (Exchange in two bases). Let $G = \mathbb{Z}_2 = \{e, s\}$, with $s^2 = e$, act on $\mathbb{C}^2$. In the standard ordered basis $b = (\mathbf{e}_1, \mathbf{e}_2)$,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the representation matrices are

$$D_b(e) = \mathbb{1}_2, \quad D_b(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now use the ordered basis $b' = (\mathbf{e}_+, \mathbf{e}_-)$, where

$$\mathbf{e}_+ = \frac{\mathbf{e}_1 + \mathbf{e}_2}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{e}_- = \frac{\mathbf{e}_1 - \mathbf{e}_2}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The basis-change matrix and the new symmetry matrix are

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D_{b'}(s) = S^{-1} D_b(s) S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices describe equivalent representations. In the double well of §7.1, they describe reflection in the localized basis ($|L\rangle, |R\rangle$) and the even/odd basis ($|+\rangle, |-\rangle$), respectively.

Let G act on quantum-state rays and suppose each $g \in G$ has a unitary implementation $U(g)$, with $U(e) = \mathbb{1}$. Then

$$g \cdot [\psi] = [U(g)\psi] \quad \forall \psi \neq 0.$$

For $g, h \in G$, the group action law gives

$$\begin{aligned} [U(g)U(h)\psi] &= g \cdot (h \cdot [\psi]) \\ &= (gh) \cdot [\psi] = [U(gh)\psi]. \end{aligned}$$

Thus $U(g)U(h)$ and $U(gh)$ implement the same ray symmetry. By Theorem 7.7, they differ by a phase independent of ψ .

Definition 7.15 (Unitary projective representation). A *unitary projective representation* assigns a unitary operator $U(g)$ to each $g \in G$, with $U(e) = \mathbb{1}$, such that

$$U(g)U(h) = c(g, h)U(gh), \quad c(g, h) \in U(1). \quad (7.6)$$

The scalar function c is its *factor system*.

unitary projective representation
么正射影表示

factor system
因子系

When $c(g, h) = 1$ for all g, h , this is a *linear representation*, as in the double well. For a central extension of G by a phase subgroup $A \subseteq U(1)$, let ρ be an ordinary unitary representation in which each $a \in A$ acts as $a\mathbb{1}$. Applying ρ to the lift multiplication law (6.15) gives

$$\rho(s(g))\rho(s(h)) = c(g, h)\rho(s(gh)).$$

Thus the chosen lifts $s(g)$ give projective operators $U(g) = \rho(s(g))$ on the same vector space.

Example 7.16 (A projective representation of rotations). The spin state space of a spin- $\frac{1}{2}$ particle is $\mathbb{C}^2$. It carries the *defining representation* of $SU(2)$, in which

defining representation
定义表示

7. Representations

each group element acts by its own 2×2 matrix. Recall from Example 2.20 that an element of $SU(2)$ has the form

$$u = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}, \quad z, w \in \mathbb{C}, \quad |z|^2 + |w|^2 = 1.$$

Its action on a spin state is

$$\rho_{\text{def}}(u) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = u \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} z\psi_1 - \bar{w}\psi_2 \\ w\psi_1 + \bar{z}\psi_2 \end{pmatrix}.$$

The representation law is $\rho_{\text{def}}(uv) = uv = \rho_{\text{def}}(u)\rho_{\text{def}}(v)$.

We now ask whether the same spin transformations can define a linear representation of $SO(3)$ on $\mathbb{C}^2$. The map $R : SU(2) \rightarrow SO(3)$ from Definition 3.15 is determined by

$$u(\vec{x} \cdot \vec{\sigma})u^\dagger = (R(u)\vec{x}) \cdot \vec{\sigma} \quad \forall \vec{x} \in \mathbb{R}^3,$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. A lift of $g \in SO(3)$ is therefore a matrix $U(g) \in SU(2)$ satisfying $R(U(g)) = g$. Choose one lift for each g , with $U(e) = \mathbb{1}_2$. Since

$$R(U(g)U(h)) = gh = R(U(gh)),$$

we have

$$U(g)U(h)U(gh)^{-1} \in \ker R = \{\pm \mathbb{1}_2\},$$

$$U(g)U(h) = c(g, h)U(gh), \quad c(g, h) \in \{\pm 1\}.$$

Thus the lifts form a projective representation of $SO(3)$.

Consider rotations through π about the first and third axes,

$$g_1 = \text{diag}(1, -1, -1), \quad g_3 = \text{diag}(-1, -1, 1).$$

They commute,

$$g_1 g_3 = g_3 g_1 = g_2 := \text{diag}(-1, 1, -1).$$

Take the matrices

$$U_1 := U(g_1) = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

$$U_3 := U(g_3) = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Their conjugation actions are

$$\begin{aligned} U_1(\vec{x} \cdot \vec{\sigma})U_1^\dagger &= \sigma_1(\vec{x} \cdot \vec{\sigma})\sigma_1 \\ &= x_1\sigma_1 - x_2\sigma_2 - x_3\sigma_3, \\ U_3(\vec{x} \cdot \vec{\sigma})U_3^\dagger &= \sigma_3(\vec{x} \cdot \vec{\sigma})\sigma_3 \\ &= -x_1\sigma_1 - x_2\sigma_2 + x_3\sigma_3. \end{aligned}$$

Comparing the coefficients of the Pauli matrices gives

$$R(U_1) = \text{diag}(1, -1, -1) = g_1, \quad R(U_3) = \text{diag}(-1, -1, 1) = g_3.$$

The opposite signs are equally valid lifts, since $R(-U_j) = R(U_j)$.

Direct multiplication gives

$$U_1 U_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2,$$

$$U_3 U_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2.$$

Taking $U(g_2) = -i\sigma_2$, the two factors are

$$c(g_1, g_3) = -1, \quad c(g_3, g_1) = 1.$$

For arbitrary phase choices $U'_1 = e^{i\alpha}U_1$ and $U'_3 = e^{i\beta}U_3$,

$$\begin{aligned} U'_1 U'_3 &= e^{i(\alpha+\beta)} U_1 U_3 \\ &= -e^{i(\alpha+\beta)} U_3 U_1 = -U'_3 U'_1. \end{aligned}$$

An ordinary representation would require

$$U'_1 U'_3 = U'(g_1 g_3) = U'(g_3 g_1) = U'_3 U'_1.$$

Hence no choice of phases makes all the factors $c(g, h)$ equal to 1. Thus $\mathbb{C}^2$ carries a linear representation of $SU(2)$ and the projective representation of $SO(3)$ constructed above, which cannot be made linear by changing phases. The map $u \mapsto R(u)$ itself gives a linear representation of $SU(2)$ on $\mathbb{R}^3$; its kernel is $\{\pm 1_2\}$, so this three-dimensional representation is not faithful.

7.2.1. Phase factors and second group cohomology

READING. [Moore, §15.3].

Associativity constrains the factor system. The two paths in the following diagram must agree. Each arrow multiplies its source by the displayed scalar phase.

$$\begin{array}{ccc} U(ghk) & \xrightarrow{c(gh,k)} & U(gh)U(k) \\ \downarrow c(g,hk) & & \downarrow c(g,h) \\ U(g)U(hk) & \xrightarrow{c(h,k)} & U(g)U(h)U(k). \end{array}$$

The operators are linear, so scalar phases pass through them unchanged. Equating the products of phases along the two paths gives the *2-cocycle condition*,

2-cocycle condition
2-余循环条件

$$c(g, h)c(gh, k) = c(h, k)c(g, hk), \quad c(e, g) = c(g, e) = 1. \quad (7.7)$$

The normalization follows from $U(e) = 1$. A function satisfying these conditions is called a *normalized 2-cocycle*.

Proposition 7.17 (The central extension of a factor system). *Let G be an abstract group and A an Abelian group. Suppose $c : G \times G \rightarrow A$ satisfies (7.7). On the set $A \times G$, the product*

$$(a, g)(b, h) = (ab c(g, h), gh)$$

7. Representations

defines a group E_c , which is a central extension of G by A . Every central extension of G by A is obtained in this form after choosing lifts of the elements of G .

If $A \subseteq U(1)$, unitary projective representations on V with factor system c correspond bijectively to ordinary unitary representations ρ_c of E_c satisfying $\rho_c(a, e) = a1_V$. The correspondence is

$$\rho_c(a, g) = aU(g), \quad U(g) = \rho_c(1, g).$$

Proof. The two groupings of a triple product agree by the cocycle condition. The identity is $(1, e)$, and

$$(a, g)^{-1} = (a^{-1}c(g, g^{-1})^{-1}, g^{-1}).$$

Taking $(g, h, k) = (g, g^{-1}, g)$ in (7.7) gives $c(g, g^{-1}) = c(g^{-1}, g)$, so this is both a left and a right inverse. Projection $(a, g) \mapsto g$ is surjective with central kernel $A \times \{e\}$, giving the exact sequence

$$1 \longrightarrow A \xrightarrow{a \mapsto (a, e)} E_c \xrightarrow{(a, g) \mapsto g} G \longrightarrow 1.$$

Conversely, consider a central extension $1 \rightarrow A \rightarrow E \xrightarrow{\pi} G \rightarrow 1$, identifying A with its kernel subgroup. As in §6.5.3, choose lifts $s(g)$ with $s(e) = e$ and define

$$c(g, h) = s(g)s(h)s(gh)^{-1} \in A.$$

Associativity and centrality give

$$\begin{aligned} (s(g)s(h))s(k) &= c(g, h)c(gh, k)s(ghk), \\ s(g)(s(h)s(k)) &= c(h, k)c(g, hk)s(ghk), \end{aligned}$$

so c obeys the cocycle condition. The normalization follows from $s(e) = e$. Every element of E has a unique expression $a s(g)$, and

$$(a s(g))(b s(h)) = ab c(g, h)s(gh).$$

Hence $(a, g) \mapsto a s(g)$ is an isomorphism $E_c \rightarrow E$ preserving the inclusion of A and the projection onto G .

For the representation correspondence, the projective law gives

$$\rho_c(a, g)\rho_c(b, h) = ab c(g, h)U(gh) = \rho_c((a, g)(b, h)).$$

Conversely, if ρ_c is a unitary representation with $\rho_c(a, e) = a1_V$, then $U(g) = \rho_c(1, g)$ satisfies

$$U(g)U(h) = \rho_c(c(g, h), gh) = c(g, h)U(gh).$$

The constructions are inverse because $(a, g) = (a, e)(1, g)$. □

For the sign-valued factor system in Example 7.16, take $G = SO(3)$ and $A = \{\pm 1\}$. The resulting extension is identified with $SU(2)$ by the group isomorphism

$$F : E_c \longrightarrow SU(2), \quad F(\varepsilon, g) = \varepsilon U(g).$$

It is bijective because the two lifts of each rotation g are $\pm U(g)$, and it preserves multiplication because

$$F(\varepsilon, g)F(\eta, h) = \varepsilon\eta c(g, h)U(gh) = F((\varepsilon, g)(\eta, h)).$$

Thus the projective spinor representation of $SO(3)$ becomes the defining linear representation of its central extension $SU(2)$, on the same space $\mathbb{C}^2$.

The phase of each $U(g)$ can be chosen independently without changing its action on rays. For $d : G \rightarrow U(1)$ with $d(e) = 1$, set $U'(g) = d(g)U(g)$. Then

$$U'(g)U'(h) = d(g)d(h)c(g, h)U(gh) = c'(g, h)U'(gh),$$

where

$$c'(g, h) = (\delta d)(g, h)c(g, h), \quad (\delta d)(g, h) = d(g)d(h)d(gh)^{-1}. \quad (7.8)$$

A factor of the form δd is called a *2-coboundary*. It is itself a normalized 2-cocycle, since

2-coboundary
2-余边界

$$(\delta d)(g, h)(\delta d)(gh, k) = \frac{d(g)d(h)d(k)}{d(ghk)} = (\delta d)(h, k)(\delta d)(g, hk).$$

Changing lifts in a central extension has exactly the same effect. For $d : G \rightarrow A$ with $d(e) = 1$, put $s'(g) = d(g)s(g)$. Centrality gives

$$s'(g)s'(h) = d(g)d(h)c(g, h)s(gh) = (\delta d)(g, h)c(g, h)s'(gh).$$

Thus the new lifts have factor system $c' = (\delta d)c$.

For a finite group G , these formulas give an algebraic classification of factor systems up to rephasing.

Definition 7.18 (Second group cohomology). Let G be a finite group. The normalized $U(1)$ -valued 2-cocycles form an Abelian group $Z^2(G, U(1))$ under pointwise multiplication,

$$(c_1 c_2)(g, h) = c_1(g, h)c_2(g, h).$$

The 2-coboundaries form a subgroup $B^2(G, U(1))$. The *second group cohomology group* is the quotient

second group cohomology group
二阶群上同调群

$$H^2(G, U(1)) = Z^2(G, U(1)) / B^2(G, U(1)).$$

Products and pointwise inverses preserve the cocycle condition. The coboundaries form a subgroup because $\delta(d_1 d_2) = (\delta d_1)(\delta d_2)$ and $\delta(d^{-1}) = (\delta d)^{-1}$. Thus a class $[c] \in H^2(G, U(1))$ consists precisely of factor systems related by rephasing.

equivalent extensions
等价扩张

7. Representations

The same quotient classifies central extensions with fixed kernel $U(1)$ and quotient G . Two such extensions, with maps ι, π and ι', π' , are *equivalent extensions* if there is a group isomorphism $F : E' \rightarrow E$ satisfying

$$F \circ \iota' = \iota, \quad \pi \circ F = \pi'.$$

Thus F preserves both the specified central phases and the quotient group elements. For $E_{c'}$ and E_c , any such map must have the form

$$F(a, g) = (a d(g), g), \quad d : G \rightarrow U(1), \quad d(e) = 1.$$

Comparing the products of the lifts gives

$$\begin{aligned} F((1, g)(1, h)) &= (c'(g, h)d(gh), gh), \\ F(1, g)F(1, h) &= (d(g)d(h)c(g, h), gh). \end{aligned}$$

Hence F is a homomorphism precisely when $c' = (\delta d)c$; its inverse sends (a, g) to $(a d(g)^{-1}, g)$. Together with Proposition 7.17, this proves the bijection

$$H^2(G, U(1)) \longleftrightarrow \left\{ \begin{array}{l} \text{central extensions of } G \text{ by } U(1) \\ \text{up to equivalence} \end{array} \right\}.$$

The identity class $[1]$ consists of those factor systems for which a rephasing makes $c'(g, h) = 1$ for every g, h . The corresponding extension splits, and the rephased operators form an ordinary representation, as in §6.5.3. For a fixed c , specifying a projective representation still requires a representation of E_c in which the kernel acts by its scalar phases. Its matrices and dimension are additional data. For example, all ordinary representations have $c = 1$, although they can be inequivalent.

Example 7.19 (The two phase-factor classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$). Write $G = \langle x, z \mid x^2 = z^2 = e, xz = zx \rangle$ and set $A = U(x)$, $B = U(z)$. Their squares are scalar phases, so rephase them to obtain $A^2 = B^2 = \mathbb{1}$. Since $xz = zx$, their products differ by a scalar phase,

$$AB = \eta BA.$$

Using $A^2 = \mathbb{1}$ gives $B = \eta ABA = \eta^2 B$, hence $\eta = \pm 1$. Rephasing A and B leaves η unchanged.

Choose the remaining representative to be $U(xz) = AB$. For $a, b \in \mathbb{Z}_2$, we then have $U(x^a z^b) = A^a B^b$, and

$$(A^a B^b)(A^{a'} B^{b'}) = \eta^{ba'} A^{a+a'} B^{b+b'}.$$

Thus every factor system is equivalent to one of $c_\eta((a, b), (a', b')) = \eta^{ba'}$. The same argument applies to lifts in E_c for any normalized 2-cocycle, so these exhaust the cohomology classes. For $\eta = +1$ the operators form an ordinary representation. For $\eta = -1$, take $A = \sigma_1$ and $B = \sigma_3$, the two-site clock and shift algebra of §6.5.2. The two classes are distinct, and the product of two copies of c_{-1} is c_{+1} . Therefore

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2.$$

The class $[c]$ has a physical consequence for energy degeneracies. Let $U(g)$ be unitary projective symmetry operators of a Hermitian Hamiltonian H , so $[H, U(g)] = 0$ for every g . Suppose an energy eigenvalue E is nondegenerate, with eigenvector ψ . Each symmetry must preserve this one-dimensional eigenspace, so

$$U(g)\psi = \lambda(g)\psi, \quad \lambda(g) \in U(1).$$

Applying $U(g)U(h) = c(g, h)U(gh)$ to ψ gives

$$\lambda(g)\lambda(h) = c(g, h)\lambda(gh), \quad c(g, h) = \frac{\lambda(g)\lambda(h)}{\lambda(gh)} = (\delta\lambda)(g, h).$$

Thus c is a coboundary and $[c] = [1]$. A nontrivial class cannot be realized by scalar symmetry actions on a single state. It requires each energy eigenspace to contain more than one linearly independent state, forcing degeneracy.

Example 7.20 (A spin doublet protected by rotation symmetry). Consider a spin- $\frac{1}{2}$ degree of freedom with symmetry under rotations through π about the first and third axes. The spatial rotations generate $\mathbb{Z}_2 \times \mathbb{Z}_2$. Up to overall phases, their operators on $\mathbb{C}^2$ are

$$A = \sigma_1, \quad B = \sigma_3, \quad A^2 = B^2 = \mathbb{1}_2, \quad AB = -BA.$$

They realize the nontrivial cohomology class in Example 7.19. A Hermitian Hamiltonian with both symmetries satisfies

$$[H, A] = [H, B] = 0.$$

These conditions mean that A and B each preserve every energy eigenspace; their actions within such an eigenspace need not commute.

Since H and A commute, choose a nonzero simultaneous eigenstate,

$$H\psi = E\psi, \quad A\psi = a\psi, \quad a = \pm 1.$$

Applying B gives

$$\begin{aligned} H(B\psi) &= B(H\psi) = E(B\psi), \\ A(B\psi) &= -B(A\psi) = -a(B\psi). \end{aligned}$$

The state $B\psi$ is nonzero because B is unitary. It has the same energy as ψ but the opposite eigenvalue of the Hermitian operator A , so the two states are orthogonal. On this two-dimensional spin space, it follows that $H = E\mathbb{1}_2$. This scalar Hamiltonian commutes with both A and B , even though A and B anticommute.

Any perturbation preserving both symmetries leaves this pairing intact. The *symmetry-protected doublet* is a physical consequence of the rephasing-invariant minus sign in $AB = -BA$. The boundary spin of the Haldane chain carries this same projective symmetry action, as discussed in §15.2.

symmetry-protected doublet
对称性保护的双重态

In Example 7.19, rephasing made the unitary operators square to the identity. For an antiunitary operator, its square is unchanged by rephasing, as time reversal illustrates below. Antiunitary symmetries conjugate scalar phases, so extending the preceding cohomology description to them requires a cocycle condition modified by that conjugation.

7. Representations

Example 7.21 (Time reversal and phase factors). For the spin- $\frac{1}{2}$ time reversal of Example 7.6, reversing time twice restores every physical state,

$$[T^2\psi] = [-\psi] = [\psi].$$

Thus time reversal on rays generates $\{e, t\} \cong \mathbb{Z}_2$, with $t^2 = e$. Its implementing operators $\mathbb{1}$ and T obey the group law up to the phase $c(t, t) = -1$, since $T^2 = -\mathbb{1}$. Rephasing T cannot remove this sign. Antiunitarity gives

$$(e^{i\theta}T)^2 = e^{i\theta}e^{-i\theta}T^2 = -\mathbb{1}.$$

The sign responsible for Kramers degeneracy is therefore independent of the phase chosen for T .

7.3. EXAMPLES OF REPRESENTATIONS

Example 7.22 (One-dimensional representations from homomorphisms). The sign homomorphism of Chapter 5 gives a one-dimensional representation,

$$\rho_{\text{sgn}} : S_n \longrightarrow GL(1, \mathbb{C}), \quad \rho_{\text{sgn}}(\sigma) = \text{sgn } \sigma.$$

Its kernel is A_n , so for $n \geq 3$ it is not faithful. Similarly,

$$\rho_{\text{det}} : GL(n, \mathbb{C}) \longrightarrow GL(1, \mathbb{C}), \quad \rho_{\text{det}}(A) = \det A$$

is a representation because $\det(AB) = \det A \det B$. These examples reinterpret familiar homomorphisms as actions on a one-dimensional vector space.

The determinant also gives a family of one-dimensional complex representations of $GL(n, \mathbb{R})$,

$$\rho_{\mu}(A) = |\det A|^{\mu} := \exp(\mu \log |\det A|), \quad \mu \in \mathbb{C}.$$

The absolute determinant is positive, so the real logarithm defines the power unambiguously. Multiplicativity follows from

$$\log |\det(AB)| = \log |\det A| + \log |\det B|.$$

Example 7.23 (Representations of a cyclic group). Let $C_N = \langle r \mid r^N = 1 \rangle$, with $N \geq 2$. A one-dimensional complex representation is determined by the scalar $z = \rho(r)$. Since $z^N = 1$, the possibilities are

$$\rho_m(r^{\ell}) = e^{-2\pi i m \ell / N}, \quad m = 0, \dots, N-1. \quad (7.9)$$

The product law follows by adding exponents. Each representation is unitary, since these scalars have modulus one. The case $m = 0$ is trivial. In one dimension a change of basis is multiplication by a nonzero scalar and leaves every representation matrix unchanged, so distinct values of m give inequivalent representations.

Example 7.24 (Scalar representations of additive groups). For the additive group $\mathbb{Z}$, choose any $z \in \mathbb{C}^*$ and put

$$\rho_z(n) = z^n, \quad n \in \mathbb{Z}.$$

The law $z^{n+m} = z^n z^m$ verifies that this is a representation. Conversely, a one-dimensional complex representation of $\mathbb{Z}$ is fixed by its value $z = \rho(1)$, which determines all powers, including $\rho(-n) = \rho(n)^{-1}$. It is unitary precisely when $|z| = 1$.

For the additive group $\mathbb{R}$, the continuous maps

$$\rho_s(x) = e^{sx}, \quad s \in \mathbb{C},$$

are representations because $e^{s(x+y)} = e^{sx} e^{sy}$. Their absolute values are $e^{\operatorname{Re}(s)x}$, so these representations are unitary precisely when $\operatorname{Re}(s) = 0$. Restricting to integer $x = n$ gives the preceding example with $z = e^s$.

Example 7.25 (Unitary representations of lattice translations). Let $G = \mathbb{Z}^d$ with addition. A one-dimensional unitary representation is determined by its values on the d integer basis vectors. Write these values as e^{-ik_a} , with $k_a \in \mathbb{R}$. Then

$$\rho_k(n) = e^{-ik \cdot n}, \quad n \in \mathbb{Z}^d. \quad (7.10)$$

Every such representation has this form, since

$$n = \sum_a n_a \mathbf{e}_a, \quad \rho_k(n) = \prod_a \rho_k(\mathbf{e}_a)^{n_a}.$$

Replacing any k_a by $k_a + 2\pi\ell_a$, with $\ell_a \in \mathbb{Z}$, gives the same representation. The phases therefore label k modulo $2\pi\mathbb{Z}^d$.

Example 7.26 (Defining representations of matrix groups). A matrix group $G \subset GL(n, \kappa)$ has its *defining representation* on κ^n , given by $\rho(g)v = gv$. It is faithful. For example, $SO(3)$ has its defining representation on $\mathbb{R}^3$, and $SU(2)$ has its defining representation on $\mathbb{C}^2$. The homomorphism $R : SU(2) \rightarrow SO(3)$ from Chapter 3 also gives a three-dimensional representation of $SU(2)$, $\rho(u)v = R(u)v$. Its kernel is $\{\pm 1_2\}$. Thus one group can act on vector spaces of different dimensions, and faithfulness depends on the representation.

Example 7.27 (Lorentz boosts in one space dimension). Write a spacetime point as $(x^0, x^1) = (ct, x)$. A *boost* with real *rapidity* θ acts by

$$\begin{pmatrix} x^{0'} \\ x^{1'} \end{pmatrix} = B(\theta) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}, \quad B(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}.$$

boost
推动
rapidity
快度

With $\eta = \operatorname{diag}(-1, 1)$, direct multiplication gives $B(\theta)^T \eta B(\theta) = \eta$, so the interval $-(x^0)^2 + (x^1)^2$ is preserved. The hyperbolic addition formulas give

$$B(\theta)B(\phi) = B(\theta + \phi), \quad B(0) = 1_2.$$

Thus $\theta \mapsto B(\theta)$ is a two-dimensional representation of $(\mathbb{R}, +)$; its image is the connected boost group $SO_0(1, 1)$.

7. Representations

In the coordinates $x^+ = x^0 + x^1$ and $x^- = x^0 - x^1$, the same transformation becomes

$$x^{+'} = e^\theta x^+, \quad x^{-'} = e^{-\theta} x^-.$$

This basis change displays two one-dimensional representations. More generally, the boost group has one-dimensional representations

$$\rho_s(B(\theta)) = e^{s\theta}, \quad s \in \mathbb{C}.$$

They obey the group law by rapidity addition and are unitary precisely when $\text{Re}(s) = 0$. The defining boost matrices instead preserve the indefinite form η . For $\theta \neq 0$, their eigenvalues $e^{\pm\theta}$ have moduli different from one, so these matrices are not unitary.

Definition 7.28 (Permutation representation). Suppose G acts on a finite set X . Let V_X have basis $\{\mathbf{e}_x : x \in X\}$, and define

$$\rho_X(g)\mathbf{e}_x = \mathbf{e}_{g \cdot x}.$$

permutation representation
置换表示

Extending linearly gives the *permutation representation* on V_X . Its dimension is $|X|$.

Indeed, $\rho_X(g)\rho_X(h)\mathbf{e}_x = \mathbf{e}_{g \cdot (h \cdot x)} = \mathbf{e}_{(gh) \cdot x}$, so the group law holds on every basis vector. In this basis all matrices are permutation matrices and are unitary for the standard inner product.

This also restates the action on functions from Definition 4.36. On the space of functions $f : X \rightarrow \kappa$, put

$$(\rho_X(g)f)(x) = f(g^{-1} \cdot x).$$

For the delta-function basis, $\rho_X(g)\delta_x = \delta_{g \cdot x}$. In particular, the column of function values transforms by the same permutation matrices.

Definition 7.29 (Left regular representation). For a finite group G , take a vector space with basis $\{\mathbf{e}_h : h \in G\}$. The *left regular representation*, denoted $L = \rho_{\text{reg}}$, is

left regular representation
左正则表示

$$L(g)\mathbf{e}_h = \mathbf{e}_{gh}.$$

It is the permutation representation of G acting on itself by left multiplication, and has dimension $|G|$.

It is faithful, since $L(g)\mathbf{e}_e = \mathbf{e}_g$ equals $\mathbf{e}_e$ only for $g = e$. This turns the left-multiplication construction in Cayley's theorem into a linear representation.

Identifying $\mathbf{e}_h$ with the function

$$\delta_h(x) = \begin{cases} 1, & x = h, \\ 0, & x \neq h, \end{cases}$$

gives the equivalent function-space formula

$$(L(g)f)(x) = f(g^{-1}x), \quad L(g)\delta_h = \delta_{gh}.$$

For complex functions, this is the finite-group case of the left regular representation on $L^2(G)$ in §8.2.

Example 7.30 (The regular representation of the Klein group). Let $G = \{e, a, b, c\}$, with $a^2 = b^2 = c^2 = e$ and $ab = c$. In the ordered basis $(\mathbf{e}_e, \mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c)$, multiplication by a exchanges e with a and b with c , giving

$$D_{\text{reg}}(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Every element's matrix is obtained from its left-multiplication action in the same way.

7.3.1. A periodic hopping chain

Example 7.31 (Fourier states on a ring). Consider $N \geq 3$ sites on a ring, labelled $j = 0, \dots, N-1$ modulo N . The amplitudes form a vector in $\mathbb{C}^N$. Write $|j\rangle$ for the basis vector with its only nonzero entry, equal to one, at site j . The bra $\langle j|$ is its adjoint row, so $|j\rangle\langle\ell|$ takes the component at site ℓ and places it at site j . Nearest-neighbour hopping with real $t > 0$ is described by

Nearest-neighbour hopping
最近邻跃迁

$$H = -t \sum_{j=0}^{N-1} (|j+1\rangle\langle j| + |j\rangle\langle j+1|).$$

Define translation by one site,

$$T|j\rangle = |j+1\rangle.$$

It is the shift matrix P from (6.11). It is unitary and satisfies $T^N = \mathbf{1}$. The operators T^ℓ therefore give a representation of C_N . Since

$$H = -t(T + T^{-1}), \quad [H, T] = 0, \quad (7.11)$$

the Hamiltonian preserves the translation eigenspaces.

Define the discrete waves

$$|k_m\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{ik_m j} |j\rangle, \quad k_m = \frac{2\pi m}{N}, \quad m = 0, \dots, N-1. \quad (7.12)$$

Translating and relabelling $\ell = j+1$ gives

$$T|k_m\rangle = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{ik_m(\ell-1)} |\ell\rangle = e^{-ik_m} |k_m\rangle.$$

The relabelling is valid at the boundary because $e^{ik_m N} = 1$. The minus sign follows from the positive exponent in (7.12) and the shift $T|j\rangle = |j+1\rangle$.

7. Representations

These N waves form an orthonormal basis,

$$\langle k_m | k_{m'} \rangle = \frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi i(m'-m)j/N} = \delta_{mm'}.$$

The finite geometric sum is zero for $m \neq m'$ and equals N for $m = m'$. Using (7.11),

$$H|k_m\rangle = -t(e^{-ik_m} + e^{ik_m})|k_m\rangle = -2t \cos k_m |k_m\rangle. \quad (7.13)$$

symmetry sector
对称性子空间

Thus each wave spans a one-dimensional *symmetry sector* on which T^ℓ acts by $e^{-ik_m\ell}$. These are precisely the representations in (7.9). The momentum label distinguishes how the state transforms under translation.

In matrix language, let $F_{jm} = e^{ik_m j} / \sqrt{N}$. Its columns are the new basis vectors, so

$$\begin{aligned} F^\dagger T F &= \text{diag}(e^{-ik_0}, \dots, e^{-ik_{N-1}}), \\ F^\dagger H F &= \text{diag}(-2t \cos k_0, \dots, -2t \cos k_{N-1}). \end{aligned}$$

The Fourier transform is the basis change that displays the translation representations. In this example every translation sector has dimension one, so the same basis diagonalizes the Hamiltonian completely.

Example 7.32 (Two sites per unit cell). Now let the ring have N unit cells with two sites A, B in each cell. Let t_1, t_2 be real hopping amplitudes within a cell and between neighbouring cells. The basis is $|j, A\rangle, |j, B\rangle$, and

$$\begin{aligned} H = - \sum_j [& t_1 (|j, A\rangle\langle j, B| + |j, B\rangle\langle j, A|) \\ & + t_2 (|j+1, A\rangle\langle j, B| + |j, B\rangle\langle j+1, A|)]. \end{aligned}$$

Translation by a whole cell acts as $T|j, \alpha\rangle = |j+1, \alpha\rangle$, where $\alpha = A, B$, and commutes with H . Define

$$|k_m, \alpha\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ik_m j} |j, \alpha\rangle.$$

Both basis vectors at momentum k_m have the translation eigenvalue e^{-ik_m} . Their span $V_m = \text{span}\{|k_m, A\rangle, |k_m, B\rangle\}$ is therefore preserved by H . Direct substitution gives

$$\begin{aligned} H|k_m, A\rangle &= -(t_1 + t_2 e^{ik_m})|k_m, B\rangle, \\ H|k_m, B\rangle &= -(t_1 + t_2 e^{-ik_m})|k_m, A\rangle. \end{aligned}$$

In the ordered basis $(|k_m, A\rangle, |k_m, B\rangle)$,

$$H(k_m) = \begin{pmatrix} 0 & -(t_1 + t_2 e^{-ik_m}) \\ -(t_1 + t_2 e^{ik_m}) & 0 \end{pmatrix}. \quad (7.14)$$

The original $2N \times 2N$ matrix becomes N separate 2×2 blocks,

$$\mathbb{C}^{2N} = \bigoplus_{m=0}^{N-1} V_m, \quad H \cong \bigoplus_{m=0}^{N-1} H(k_m).$$

The direct-sum symbol means that every state splits uniquely into components in the spaces V_m ; §7.4 defines this construction generally. Translation fixes the momentum label but allows mixing of the two amplitudes within each sector. Symmetry has reduced the matrix problem before the remaining 2×2 matrices are diagonalized.

7.4. CONSTRUCTIONS OF REPRESENTATIONS

Definition 7.33 (Direct sum). Let (V, ρ_V) and (W, ρ_W) be representations of the same group G . The vector space $V \oplus W$ consists of pairs (v, w) with $v \in V$ and $w \in W$, with addition and scalar multiplication in each component. Their *direct-sum representation* is given by

$$(\rho_V \oplus \rho_W)(g)(v, w) = (\rho_V(g)v, \rho_W(g)w).$$

Its dimension is $\dim V + \dim W$.

If $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ and $(\mathbf{f}_1, \dots, \mathbf{f}_m)$ are bases of V and W , the combined ordered basis is

$$((\mathbf{e}_1, 0), \dots, (\mathbf{e}_n, 0), (0, \mathbf{f}_1), \dots, (0, \mathbf{f}_m)).$$

In this basis,

$$D_{V \oplus W}(g) = \begin{pmatrix} D_V(g) & 0 \\ 0 & D_W(g) \end{pmatrix}.$$

Multiplying two such matrices gives the group law separately in each block. In Example 7.14, the exchange representation is the direct sum of two one-dimensional representations, $\rho_+(s) = 1$ and $\rho_-(s) = -1$.

The *tensor product* $V \otimes W$ is spanned by symbols $v \otimes w$ subject to bilinearity,

$$\begin{aligned} (av + bv') \otimes w &= a(v \otimes w) + b(v' \otimes w), \\ v \otimes (aw + bw') &= a(v \otimes w) + b(v \otimes w'). \end{aligned}$$

If $(\mathbf{e}_i)$ and $(\mathbf{f}_a)$ are bases, the nm vectors $\mathbf{e}_i \otimes \mathbf{f}_a$ are a basis of $V \otimes W$. A general element is $\sum_{i,a} c_{ia} \mathbf{e}_i \otimes \mathbf{f}_a$.

If V and W carry inner products, their *tensor-product inner product* is defined by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle,$$

extended sesquilinearly. Orthonormal bases of V and W therefore give an orthonormal tensor-product basis.

Definition 7.34 (Tensor product representation). The *tensor product* of (V, ρ_V) and (W, ρ_W) acts by

$$(\rho_V \otimes \rho_W)(g)(v \otimes w) = \rho_V(g)v \otimes \rho_W(g)w,$$

extended linearly to $V \otimes W$. Its dimension is $\dim V \dim W$.

READING. [Moore, §11.2, pp. 227–229].

direct sum
直和

tensor product
张量积

tensor-product inner product
张量积内积

7. Representations

Applying h and then g to $v \otimes w$ gives $\rho_V(gh)v \otimes \rho_W(gh)w$, which verifies the representation law. Order the basis as $(\mathbf{e}_1 \otimes \mathbf{f}_1, \dots, \mathbf{e}_1 \otimes \mathbf{f}_m, \mathbf{e}_2 \otimes \mathbf{f}_1, \dots, \mathbf{e}_n \otimes \mathbf{f}_m)$. The matrix entries are

$$D_{V \otimes W}(g)_{ia,jb} = D_V(g)_{ij} D_W(g)_{ab}.$$

This is the *Kronecker product* of the two matrices.

Example 7.35 (Ordering a tensor-product basis). Let the nonidentity element s of $\mathbb{Z}_2$ act on two two-dimensional spaces by

$$D_V(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D_W(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the basis $(\mathbf{e}_1 \otimes \mathbf{f}_1, \mathbf{e}_1 \otimes \mathbf{f}_2, \mathbf{e}_2 \otimes \mathbf{f}_1, \mathbf{e}_2 \otimes \mathbf{f}_2)$, their tensor product is

$$D(s) = D_V(s) \otimes D_W(s) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

For instance, s sends $\mathbf{e}_1 \otimes \mathbf{f}_2$ to $-\mathbf{e}_2 \otimes \mathbf{f}_2$, giving the second column. Also $D(s)^2 = D_V(s)^2 \otimes D_W(s)^2 = \mathbf{1}_4$, as required by $s^2 = e$.

Order the same basis instead as $(\mathbf{e}_1 \otimes \mathbf{f}_1, \mathbf{e}_2 \otimes \mathbf{f}_1, \mathbf{e}_1 \otimes \mathbf{f}_2, \mathbf{e}_2 \otimes \mathbf{f}_2)$. The matrix becomes

$$D'(s) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The new basis exchanges the second and third vectors, so

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D'(s) = S^{-1} D(s) S.$$

These are two matrices for the same tensor-product action. Specifying the order of the basis is therefore part of specifying its matrix.

Example 7.36 (Cell and site labels). For the two-site cells in Example 7.32, the state space is $\mathbb{C}^N \otimes \mathbb{C}^2$, with one factor for the cell label and one for the site A or B . Translation acts as $T \otimes \mathbf{1}_2$. Changing the cell factor to the Fourier basis produces the direct-sum description $\bigoplus_m V_m$. Thus tensor products describe the two labels of a state, while direct sums describe the separate momentum sectors.

Definition 7.37 (Restriction). If $L \subset G$ is a subgroup, the *restriction* of (V, ρ) to L is the representation on the same space V ,

$$\rho|_L : L \longrightarrow GL(V), \quad \ell \longmapsto \rho(\ell).$$

Kronecker product
克罗内克积

restriction
限制

The same matrices are used, with only the subgroup elements allowed.

Example 7.38 (Restricting rotations to one axis). Restrict the defining representation of $SO(3)$ on $\mathbb{R}^3$ to the $SO(2)$ subgroup of rotations about the third axis. Its matrices are

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The third-axis component is fixed, and the first two components rotate in their plane. This is a direct sum of a two-dimensional rotation representation and a one-dimensional trivial representation of $SO(2)$.

The *dual space* is $V^\vee = \text{Hom}_\kappa(V, \kappa)$, the space of *linear functionals* $\ell : V \rightarrow \kappa$. If $(\mathbf{e}_i)$ is a basis of V , its *dual basis* $(\mathbf{e}^i)$ satisfies $\mathbf{e}^i(\mathbf{e}_j) = \delta_{ij}$.

dual space
对偶空间
linear functionals
线性泛函

Definition 7.39 (Dual representation). The *dual representation* on $V^\vee$ is

dual basis
对偶基
dual representation
对偶表示

$$(\rho^\vee(g)\ell)(v) = \ell(\rho(g^{-1})v).$$

The inverse ensures that the evaluation pairing is invariant,

$$(\rho^\vee(g)\ell)(\rho(g)v) = \ell(v).$$

It also gives the correct composition order,

$$\rho^\vee(g)\rho^\vee(h)\ell = \ell \circ \rho(h^{-1}) \circ \rho(g^{-1}) = \ell \circ \rho((gh)^{-1}).$$

If a functional has coordinate column c , then $\ell(v) = c^T[v]_b$. Its transformed coordinates are $D(g^{-1})^T c$. Thus in the dual basis

$$D_{V^\vee}(g) = D_V(g^{-1})^T = D_V(g)^{-T}. \quad (7.15)$$

Definition 7.40 (Conjugate representation). For a complex vector space V , the *conjugate space* $\bar{V}$ consists of symbols $\bar{v}$, with addition and scalar multiplication defined by

conjugate space
共轭空间

$$\bar{v} + \bar{w} = \overline{v + w}, \quad a\bar{v} = \overline{\bar{a}v} \quad (a \in \mathbb{C}).$$

conjugate representation
共轭表示

The map $v \mapsto \bar{v}$ is therefore conjugate-linear. The *conjugate representation* of (V, ρ_V) is the linear action

$$\rho_{\bar{V}}(g)\bar{v} = \overline{\rho_V(g)v}.$$

If $(\mathbf{e}_i)$ is a basis of V , then $(\bar{\mathbf{e}}_i)$ is a basis of $\bar{V}$, and the matrices satisfy

$$D_{\bar{V}}(g) = \overline{D_V(g)}.$$

The bar means entrywise complex conjugation.

7. Representations

Conjugating the matrix equation $D_V(gh) = D_V(g)D_V(h)$ verifies the representation law. For unitary matrices in an orthonormal basis,

$$D_V(g)^{-T} = \overline{D_V(g)},$$

so the dual and conjugate representations have the same matrices in the corresponding bases. The evaluation pairing uses the dual space; the Hermitian inner product relates it to the conjugate space.

The representations on V and W induce an action on the space $\text{Hom}(V, W)$ of linear maps,

$$g \cdot A = \rho_W(g)A\rho_V(g)^{-1}. \quad (7.16)$$

Applying h and then g gives

$$g \cdot (h \cdot A) = \rho_W(gh)A\rho_V(gh)^{-1},$$

so this is a linear group action. The fixed operators satisfy $\rho_W(g)A\rho_V(g)^{-1} = A$, or equivalently $\rho_W(g)A = A\rho_V(g)$. They are precisely the intertwiners. For $V = W$, this is conjugation on $\text{End}(V)$, and an invariant Hamiltonian is a fixed vector of this operator representation.

The map sending $w \otimes \ell$ to the operator $v \mapsto \ell(v)w$ gives the identification

$$\text{Hom}(V, W) \cong W \otimes V^\vee.$$

Indeed, $\mathbf{f}_a \otimes \mathbf{e}^i$ corresponds to the matrix unit taking $\mathbf{e}_i$ to $\mathbf{f}_a$ and every other basis vector to zero. The tensor-product action reproduces (7.16), since the dual acts on the input with $\rho_V(g)^{-1}$.

invariant vector
不变向量

Definition 7.41 (Invariant vector and tensor). An *invariant vector* in (V, ρ) satisfies $\rho(g)v = v$ for every $g \in G$. The invariant subspace is

$$V^G = \{v \in V : \rho(g)v = v \ \forall g \in G\}.$$

invariant tensor
不变张量

An *invariant tensor* is an invariant vector in a tensor-product representation.

Example 7.42 (The invariant Euclidean tensor). Let $SO(3)$ act on $\mathbb{R}^3$ in its defining representation. In the standard orthonormal basis, consider the tensor

$$t = \sum_i \mathbf{e}_i \otimes \mathbf{e}_i, \quad t_{ij} = \delta_{ij}.$$

For $R \in SO(3)$, its transformed components are

$$\sum_i R_{ji}R_{ki} = (RR^T)_{jk} = \delta_{jk},$$

so t is invariant. This tensor expresses the rotationally invariant Euclidean inner product.

Recall the index exchange and projectors from §5.5. On $V \otimes V$, define

$$\tau(v \otimes w) = w \otimes v, \quad \Pi_{\pm} = \frac{1}{2}(1 \pm \tau).$$

Since $\tau^2 = 1$, these operators satisfy $\Pi_{\pm}^2 = \Pi_{\pm}$, $\Pi_+ \Pi_- = 0$, and $\Pi_+ + \Pi_- = 1$. The two factors carry the same representation, so τ commutes with $\rho(g) \otimes \rho(g)$. Each projector's image is therefore preserved by G .

Definition 7.43 (Symmetric and exterior squares). The *symmetric square* and *exterior square* are

symmetric square
对称平方
exterior square
外平方

$$\text{Sym}^2 V = \text{im } \Pi_+, \quad \bigwedge^2 V = \text{im } \Pi_-.$$

Restricting each operator $\rho(g) \otimes \rho(g)$ to these invariant subspaces gives their representations, and

$$V \otimes V = \text{Sym}^2 V \oplus \bigwedge^2 V.$$

If $\dim V = n$, the symmetric tensors have n diagonal components and one component for each of the $n(n-1)/2$ unordered pairs of distinct indices. Antisymmetric tensors have zero diagonal and one component for each such pair. Hence

$$\dim \text{Sym}^2 V = \frac{n(n+1)}{2}, \quad \dim \bigwedge^2 V = \frac{n(n-1)}{2}.$$

This is the algebra of symmetric and antisymmetric amplitudes under exchange of two tensor factors, and of symmetric and antisymmetric rank-two tensors under a common rotation of both indices.

For individual vectors we use the wedge notation

$$v \wedge w = \frac{v \otimes w - w \otimes v}{\sqrt{2}}.$$

If v, w are orthonormal, this antisymmetric vector has norm one. The space $\bigwedge^2 V$ is spanned by these wedges; a general element is a linear combination of them.

7.5. CHARACTERS

READING. [Moore, §11.3, pp. 230–231].

Representation matrices depend on the basis. Their traces are independent of the basis and define functions on the group.

character
特征标

Definition 7.44 (Character). The *character* of a finite-dimensional representation (V, ρ) is the function

$$\chi_V : G \longrightarrow \mathbb{C}, \quad \chi_V(g) = \text{Tr}_V \rho(g) = \text{Tr } D_V(g).$$

7. Representations

Proposition 7.45 (Elementary character identities). *The character is independent of the basis, and equivalent representations have the same character. It is a class function,*

$$\chi_V(hgh^{-1}) = \chi_V(g) \quad \forall g, h \in G,$$

and satisfies $\chi_V(e) = \dim V$. For two representations of the same group,

$$\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g), \quad \chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g). \quad (7.17)$$

For the dual and conjugate representations,

$$\chi_{V^\vee}(g) = \chi_V(g^{-1}), \quad \chi_{\overline{V}}(g) = \overline{\chi_V(g)}.$$

For a unitary representation, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

Proof. Trace is unchanged by similarity, so the basis-change and equivalence formulas imply the first claims. For conjugation,

$$\mathrm{Tr} \rho(hgh^{-1}) = \mathrm{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{Tr} \rho(g).$$

Thus the character is a class function in the sense of Definition 4.29. At e , the operator is $\mathbb{1}_V$, whose trace is $\dim V$. The trace of a block-diagonal matrix is the sum of the block traces. For the tensor product,

$$\mathrm{Tr}(D_V \otimes D_W) = \sum_{i,a} (D_V)_{ii} (D_W)_{aa} = \mathrm{Tr} D_V \mathrm{Tr} D_W.$$

The dual and conjugate formulas follow from $D_{V^\vee}(g) = D_V(g^{-1})^T$ and $D_{\overline{V}}(g) = \overline{D_V(g)}$. For unitary matrices, $D_V(g^{-1}) = D_V(g)^\dagger$, whose trace is the complex conjugate of $\mathrm{Tr} D_V(g)$. $\square$

A one-dimensional representation has $\chi(g) = \rho(g)$, since its matrix has one entry. Thus the cyclic and lattice representations in (7.9) and (7.10) are also called unitary characters. The trivial representation has constant character $\dim V$.

Example 7.46 (Fixed points and the regular character). For a permutation representation on X , a diagonal entry is one exactly when the corresponding point is fixed. Therefore

$$\chi_X(g) = |\mathrm{Fix}_X(g)|.$$

In the regular representation, left multiplication by g fixes h exactly when $gh = h$, which forces $g = e$. Hence

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G|, & g = e, \\ 0, & g \neq e. \end{cases}$$

Example 7.47 (Permutation characters for S_3). Let S_3 permute the basis vectors of κ^3 . Its character equals three at the identity, one at a transposition, and zero at a three-cycle. The vector $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ is fixed, and the plane

$$W = \{(v_1, v_2, v_3) : v_1 + v_2 + v_3 = 0\}$$

is preserved by permutations. Thus $\kappa^3 = \kappa(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \oplus W$, and the direct-sum formula gives $\chi_W = \chi_{\mathrm{perm}} - 1$. All the resulting characters can be listed by conjugacy-class representatives,

Representation	e	$(1\ 2)$	$(1\ 2\ 3)$
Trivial	1	1	1
Sign	1	-1	1
Permutation on κ^3	3	1	0
W	2	0	-1

For example, the entry $\chi_W(e) = 2$ records the dimension of the plane.

Example 7.48 (Equal characters without equivalence). The two representations of $\mathbb{Z}$ given by

$$\rho_0(n) = \mathbb{1}_2, \quad \rho_1(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

both have character $\chi(n) = 2$ for every n . They are inequivalent, since conjugating $\rho_0(1) = \mathbb{1}_2$ always gives $\mathbb{1}_2$, whereas $\rho_1(1) \neq \mathbb{1}_2$. For finite-dimensional complex representations of finite groups, or continuous representations of compact groups, equal characters do imply equivalence by Corollary 9.20.

The exchange projectors of §7.4 also determine the characters of symmetric and exterior squares. Let $D = D_V(g)$. Since $\Pi_{\pm}$ project onto invariant summands, the trace on those summands is

$$\chi_{\text{Sym}^2 V}(g) = \text{Tr}((D \otimes D)\Pi_+), \quad \chi_{\wedge^2 V}(g) = \text{Tr}((D \otimes D)\Pi_-).$$

We already have $\text{Tr}(D \otimes D) = \chi_V(g)^2$. In the basis $(\mathbf{e}_i \otimes \mathbf{e}_j)$, the diagonal coefficient of $(D \otimes D)\tau$ at $\mathbf{e}_i \otimes \mathbf{e}_j$ is $D_{ij}D_{ji}$. Consequently

$$\text{Tr}((D \otimes D)\tau) = \sum_{i,j} D_{ij}D_{ji} = \text{Tr}(D^2) = \chi_V(g^2).$$

Substituting $\Pi_{\pm} = (\mathbb{1} \pm \tau)/2$ yields

$$\begin{aligned} \chi_{\text{Sym}^2 V}(g) &= \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}, \\ \chi_{\wedge^2 V}(g) &= \frac{\chi_V(g)^2 - \chi_V(g^2)}{2}. \end{aligned} \tag{7.18}$$

At $g = e$, these give $n(n+1)/2$ and $n(n-1)/2$, agreeing with the dimensions obtained by counting tensor components.

KEY POINTS

- Wigner’s theorem implements transition-probability-preserving symmetries of rays by unitary or antiunitary operators. For an antiunitary operator, $A = UK$ and $A^2 = U\bar{U}$ in an orthonormal basis.
- A linear representation is a homomorphism $G \rightarrow GL(V)$. A basis gives matrices; changing the basis conjugates all of them by the same matrix.

7. Representations

- Projective multiplication has phase factors satisfying the 2-cocycle condition. A projective representation becomes an ordinary representation of E_c , with central phases acting as scalar operators. For finite G , $H^2(G, U(1))$ classifies central extensions by $U(1)$ up to equivalence; a class can support different representations.
- Intertwiners respect the group action. An invertible intertwiner gives an equivalence of representations; invariant Hamiltonians intertwine a representation with itself.
- Translation symmetry separates momentum sectors. One amplitude per cell gives one-dimensional sectors; two amplitudes give two-dimensional Hamiltonian blocks.
- Direct sums, tensor products, restrictions, duals, and conjugates construct representations from known ones. Operators carry the action in (7.16).
- A character is a basis-independent class function given by the trace. Direct sums add characters, tensor products multiply them, and symmetric and exterior squares satisfy (7.18).

KEY TERMS. representation 表示 · faithful representation 忠实表示 · equivalent representations 等价表示 · intertwiner 交织算符 · projective representation 射影表示 · factor system 因子系 · 2-cocycle 2-余循环 · 2-coboundary 2-余边界 · group cohomology 群上同调 · permutation representation 置换表示 · regular representation 正则表示 · character 特征标 · tensor product 张量积 · dual representation 对偶表示 · conjugate representation 共轭表示 · symmetric square 对称平方 · exterior square 外平方 · antiunitary operator 反么正算符

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