Part I.

Groups

1. Introduction

1.1. Three themes of twentieth-century physics

Reading. [Yang].

quantization
量子化

symmetry
对称性

phase factor
相因子

In a lecture at UNESCO in Paris in July 2002, C. N. Yang named *quantization*, *symmetry* and the *phase factor* as the "three thematic melodies of 20th century theoretical physics" [Yang].

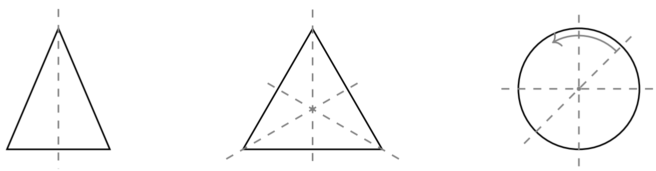
Melody	Where it leads	Its signature
quantization	quantum mechanics	h
symmetry	group theory	G
phase factor	gauge theory, electrodynamics	$e^{i\theta}$

Quantization is central to quantum mechanics; phase factors connect quantum mechanics with electromagnetism and gauge theory. This course is about the second.

1.2. What symmetry is, and how to measure it

Some objects are more symmetric than others. An equilateral triangle is more symmetric than an isosceles one, a regular hexagon more symmetric than a regular pentagon, a circle more symmetric than any polygon. This impression can be made into a number.

Restrict attention to rigid motions of the plane, reflections included, and ask which of them leave a figure looking as it did before. A non-equilateral isosceles triangle admits two, the identity E and the reflection m across the median. An equilateral triangle admits six, the identity, reflections in the three medians, and rotations through $2\pi/3$ and $4\pi/3$. A regular n-gon admits $2n$. A circle admits infinitely many, one rotation for every angle and one reflection in every diameter.



$\{E, m\}$ 2 elements

$\{E, m_1, m_2, m_3, R_1, R_2\}$ 6 elements

$\{R(\theta)\} \cup \{\text{reflections}\}$ infinitely many

group
群

The transformations leaving an object invariant form a *group*. For a finite group its size counts the symmetry operations available. More generally symmetry is characterised by the structure of the group and not by its size alone.

The organising slogan is

$$\text{group} \longleftrightarrow \text{symmetry} \longleftrightarrow \text{conservation},$$

which is a guide rather than a chain of equivalences, since each arrow carries its own hypotheses. Stated carefully, symmetry *can* enforce degeneracies and *can* forbid transitions, and for continuous symmetries of the action it yields conserved quantities.

Group theory entered quantum mechanics through Weyl and Wigner. Weyl's *Gruppentheorie und Quantenmechanik* appeared in 1928, Wigner's *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* in 1931, translated in 1959 as *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*. Wigner's theorem is treated in Chapter 8.

1.3. Discrete symmetry

Return to the isosceles triangle and its reflection m. Reflecting twice restores the original labelling,

$$m^2 = E,$$

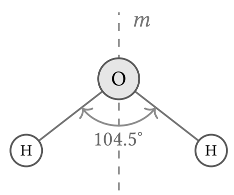
so the two operations close on themselves and form a group of order two,

$$G_1 = \{E, m\}.$$

This is the smallest group beyond the trivial one. We shall meet the same two-element structure in a molecule and in particle exchange. In each case, we ask how a physical pattern changes under the operation.

1.3.1. Mirror symmetry in a molecule

Example 1.1 (The two O–H bonds in water). At equilibrium, a water molecule has two equal O–H bonds with an angle of about 104.5° between them. Consider the plane through the oxygen atom that bisects this angle and stands perpendicular to the plane of the molecule. Its intersection with the page is the dashed line below. Reflection m in this plane exchanges the two hydrogen positions and leaves the oxygen in place. Since the two hydrogen atoms are identical, the reflected arrangement is indistinguishable from the original one.



Reflecting twice puts each atom back in its starting position, so again $m^2 = E$. The operations $\{E, m\}$ form the same two-element group as for the isosceles triangle. Water has other symmetries too in the three-dimensional space; here we examine just this one mirror operation restricted to the molecular plane.

What does this tell us about molecular vibrations? Let q_1 and q_2 be small changes in the left and right O–H bond lengths, positive for stretching and negative for compression. Reflection exchanges them,

$$m : (q_1, q_2) \longmapsto (q_2, q_1).$$

There are two particularly simple patterns. If both bonds stretch by the same amount, (q, q), reflection leaves the pattern unchanged. If one stretches while the other compresses by the same amount, $(q, -q)$, reflection gives $(-q, q)$, the negative of the original pattern. We call these patterns *symmetric* (even) and *antisymmetric* (odd) under m:

$$m(q, q) = +(q, q), \qquad m(q, -q) = -(q, -q).$$

The minus sign reverses stretching and compression; it does not mean that a bond length becomes negative.

Any pair of bond-length changes is a sum of these two patterns,

$$(q_1, q_2) = \frac{q_1 + q_2}{2}(1, 1) + \frac{q_1 - q_2}{2}(1, -1).$$

Thus symmetry organises the stretching motions before we calculate any vibration frequencies. It does not determine those frequencies, and a full description of water's vibrations also includes changes in the bond angle. Chapter 11 develops this use of molecular symmetry.

1.3.2. Particle exchange

For the quantum mechanical wave function of two identical particles, let P exchange the particle coordinates,

$$P\,|\psi(\vec{r}_1, \vec{r}_2)\rangle = |\psi(\vec{r}_2, \vec{r}_1)\rangle\,, \qquad P^2 = E,$$

so $\{E, P\}$ is once more the same two-element group.

In quantum mechanics identical *bosons* have symmetric wavefunctions and identical *fermions* antisymmetric ones.

bosons
玻色子

fermions
费米子

1.3.3. Crystals

In three-dimensional periodic crystals the crystallographic *point groups* fall into 32 types and the full spatial symmetries into 230 *space-group* types. Space groups contain the lattice translations and may contain screw rotations and glide reflections, so a space group is more than a point group with translations added. From this classification follow

point groups
点群

space-group
空间群

selection rules
跃迁选择定则

- constraints on the degeneracies of energy levels, and, together with the transformation properties of the transition operator, the *selection rules* governing transitions;
- the classification of lattice vibrations, hence the structure of phonon spectra measured by, e.g., infrared and Raman spectroscopy.

Chapter 11 takes this up.

The triangle's reflection, the chosen mirror of water, and particle exchange all gave the same group,

$$\{E, m\} \;\cong\; \{E, P\} \;\cong\; \mathbb{Z}_2 = \{1, -1\} \;\cong\; S_2.$$

Recognising this is what saved the work in each case, and Chapter 3 makes the word "same" precise.

1.4. Continuous symmetry and Noether's theorem

The reflection and permutation groups above are discrete. The rotations of a circle are not, and for such continuous families there is a stronger statement, due to Emmy Noether.

Theorem 1.2 (Noether, 1918)**.** *A continuous symmetry of the action gives rise to a conserved quantity.*

We shall not prove this. It belongs to analytical mechanics, and it needs a Lagrangian rather than only a geometrical symmetry. The standard cases are the following.

Symmetry of the action	Conserved quantity	Generator
time translation	energy	H
spatial translation	momentum	$\vec{p}$
rotation	total angular momentum	$\vec{J}$

The third column anticipates a later point, that in quantum mechanics a continuous symmetry of the dynamics is *generated* by an operator which is the conserved quantity.

Lie groups
李群

The continuous symmetry groups met here are *Lie groups*, smooth manifolds whose multiplication and inversion are smooth. The *dimension* of such a group is the number of independent real parameters needed to specify an element locally, with $\dim SU(N) = N^2 - 1$ and $\dim U(1) = 1$.

local gauge symmetry
定域规范对称性

Spin is governed by $SU(2)$ and its relation to $SO(3)$. The Standard Model rests on the *local gauge symmetry*

$$U(1) \times SU(2) \times SU(3),$$

local because the group element is chosen independently at each point of spacetime. There is one gauge field per group dimension,

$$\underset{SU(3)}{8} + \underset{SU(2)}{3} + \underset{U(1)}{1} = 12.$$

The eight of $SU(3)$ are the gluons of the strong interaction. The remaining four are electroweak, mixing into $W^{\pm}$, Z^0 and the photon.

This course does not develop Lie theory systematically. We use specific examples where they are needed and otherwise stay with finite groups.

Key terms. quantization 量子化 · symmetry 对称性 · phase factor 相因子 · group 群 · conservation 守恒 · boson 玻色子 · fermion 费米子 · point group 点群 · space group 空间群 · Lie group 李群 · Noether's theorem 诺特定理

References for this chapter

[Yang] C. N. Yang. "Thematic Melodies of Twentieth Century Theoretical Physics: Quantization, Symmetry and Phase Factor". In: *International Journal of Modern Physics A* 18 (2003). Lecture delivered at UNESCO, Paris, July 2002 (TH-2002), p. 3263.