Let $V = \mathbb{R}^2$ carry the representation of S_3 given by

$$\rho(r) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad r = (1\,2\,3), \quad s = (1\,2).$$

Both problems use this representation. Write (e_1, e_2) for the standard orthonormal basis and (e^1, e^2) for its dual basis.

Problem 24

(1) Find all real 2×2 matrices A satisfying $A\rho(r) = \rho(r)A$ and $A\rho(s) = \rho(s)A$. Hence determine $\mathrm{End}_{S_3}(V)$ by direct calculation.

(2) Write the matrices of r and s on $V^\vee$ in the dual basis. Using the identification

$$\Phi : V \otimes V^\vee \longrightarrow \mathrm{End}(V), \qquad \Phi(v \otimes \ell)(x) = \ell(x)v,$$

translate your answer to part (1) into a basis of the invariant subspace $(V \otimes V^\vee)^{S_3}$.

Problem 25

Use the ordered basis (u_0, u_1, u_2) of $\mathrm{Sym}^2 V$, where

$$u_0 = \frac{e_1 \otimes e_1 + e_2 \otimes e_2}{\sqrt{2}}, \qquad u_1 = \frac{e_1 \otimes e_1 - e_2 \otimes e_2}{\sqrt{2}},$$
$$u_2 = \frac{e_1 \otimes e_2 + e_2 \otimes e_1}{\sqrt{2}},$$

and the basis $a = (e_1 \otimes e_2 - e_2 \otimes e_1)/\sqrt{2}$ of $\bigwedge^2 V$.

(1) Write the representation matrices of r and s on $\mathrm{Sym}^2 V$ and on $\bigwedge^2 V$, by applying $\rho(g) \otimes \rho(g)$ to the displayed basis vectors.

(2) Make a table of χ_V, $\chi_{\mathrm{Sym}^2 V}$ and $\chi_{\bigwedge^2 V}$ at e, r, s by taking traces. Check the results using

$$\chi_{\mathrm{Sym}^2 V}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2},$$
$$\chi_{\bigwedge^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2}.$$