Problem 21

For each of the following one-dimensional complex representations, find its kernel and give necessary and sufficient conditions for it to be faithful and for it to be unitary.

(1) Let $C_N = \langle r \mid r^N = e \rangle$, where $N \geq 2$. For $m = 0, \ldots, N-1$, consider

$$\rho_m(r^\ell) = e^{-2\pi i m \ell / N}.$$

Express the faithfulness condition in terms of $\gcd(m, N)$.

(2) For the additive group $\mathbb{Z}$, let $z \in \mathbb{C}^*$ and

$$\rho_z(n) = z^n.$$

Treat separately the cases where z is a root of unity and where it is not. In the unitary case, write $z = e^{2\pi i \alpha}$ and express the faithfulness condition in terms of α.

(3) For the additive group $\mathbb{R}$, let $s \in \mathbb{C}$ and

$$\rho_s(x) = e^{sx}.$$

Include $s = 0$ in your analysis. Which of $\mathbb{Z}$ and $\mathbb{R}$ admits a faithful one-dimensional unitary representation?

Problem 22

Let $C_N = \langle r \mid r^N = e \rangle$, where $N \geq 2$, and consider an arbitrary central extension

$$1 \longrightarrow U(1) \xrightarrow{\iota} E \xrightarrow{\pi} C_N \longrightarrow 1.$$

Identify each phase $a \in U(1)$ with its image $\iota(a) \in E$. Choose a lift $x \in E$ of the generator, so that $\pi(x) = r$.

(1) Show that every element of E has a unique expression ax^j, with $a \in U(1)$ and $j \in \{0, \ldots, N-1\}$. Deduce that E is Abelian.

(2) Show that $x^N \in U(1)$. Find a phase $\lambda \in U(1)$ such that $y = \lambda x$ satisfies $y^N = 1$. Prove that

$$s : C_N \longrightarrow E, \qquad s(r^j) = y^j,$$

is a well-defined homomorphism with $\pi \circ s = \mathrm{id}_{C_N}$. Hence the extension splits.

(3) Use the classification of central extensions to conclude that

$$H^2(C_N, U(1)) = \{[1]\}.$$

What does this imply about removing the phase factors of a unitary projective representation of C_N?

(4) Recall the nonsplit sequence from the lecture,

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota_0} \mathbb{Z}_4 \xrightarrow{\pi_0} \mathbb{Z}_2 \longrightarrow 0,$$

where $\iota_0(a) = 2a \pmod 4$ and $\pi_0(b) = b \pmod 2$. Explain why it does not contradict your result above. Which step fails when the kernel $U(1)$ is replaced by $\mathbb{Z}_2$?

Problem 23

Consider one particle on a periodic ring of $2N$ sites, where $N \geq 2$. The states $|j\rangle$, $j = 0, \ldots, 2N-1$, are orthonormal, and site indices are taken modulo $2N$. Let

$$H_0 = -t \sum_{j=0}^{2N-1} \big(|j+1\rangle\langle j| + |j\rangle\langle j+1| \big),$$
$$V = \Delta \sum_{j=0}^{2N-1} (-1)^j |j\rangle\langle j|, \qquad H = H_0 + V,$$

where $t > 0$ and $\Delta \neq 0$ are real. Translation by one site is $T|j\rangle = |j+1\rangle$. Define the momentum states

$$|q_m\rangle = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} e^{\mathrm{i} q_m j} |j\rangle, \qquad q_m = \frac{\pi m}{N}, \quad m = 0, \ldots, 2N-1.$$

You may use the lecture result $H_0|q_m\rangle = -2t \cos q_m \, |q_m\rangle$.

(1) Determine whether H commutes with T and with T^2. Identify the translation subgroup that remains a symmetry.

(2) Calculate $\langle q_{m'}|V|q_m\rangle$. Show that V couples only momenta differing by π modulo 2π, and that H preserves each space

$$V_m = \mathrm{span}\{|q_m\rangle, |q_{m+N}\rangle\}, \qquad m = 0, \ldots, N-1.$$

(3) Write the 2×2 Hamiltonian matrix on V_m in the displayed ordered basis and find its two energy eigenvalues. Thus reduce the original $2N \times 2N$ Hamiltonian to N independent blocks.

(4) Find the matrix of T^2 on V_m. Explain why this symmetry allows the two momentum components to mix. Does the two-dimensionality of V_m require an energy degeneracy?