

Problem 13

What is the smallest symmetric group S_n into which the dihedral group D_3 can be embedded? Construct an embedding and conclude that $D_3 \cong S_3$.

Problem 14

The permutation $\phi \in S_n$ reverses the order of $\{1, 2, \dots, n\}$, so that $\phi(i) = n + 1 - i$.

- (1) Write down its cycle decomposition.
- (2) Determine whether it is even or odd for each n .
- (3) Express it as a product of the adjacent transpositions $s_i = (i \ i+1)$, where $1 \leq i < n$.

Products of permutations act from right to left.

Problem 15

For $g_1, g_2 \in G$, define the *group commutator* by

$$[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}.$$

The *commutator subgroup* $[G, G]$ is the subgroup generated by all these commutators.

- (1) Show that $[G, G] \triangleleft G$.
- (2) Let $H \triangleleft G$. Show that G/H is Abelian if and only if $[G, G] \subseteq H$.

Problem 16

Let G act on a set X , with associated homomorphism

$$\phi : G \longrightarrow S_X, \quad \phi(g)(x) = g \cdot x,$$

where S_X is the group of bijections of X .

- (1) Show that the action is effective if and only if ϕ is injective.
- (2) Show that the elements acting as the identity on all of X form a normal subgroup

$$H = \{g \in G : g \cdot x = x \text{ for all } x \in X\} \triangleleft G.$$

- (3) Show that $(gH) \cdot x = g \cdot x$ defines a well-defined, effective action of G/H on X .

Problem 17

Let

$$D = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : |z| = 1 \right\} \subset SU(2), \quad N = N_{SU(2)}(D).$$

You may use the results of Problem 11 in HW02.

- (1) Compute the quotient group N/D .
- (2) Show that conjugation by an element of N acts on D by a permutation of its two diagonal entries, and that this permutation depends only on the coset in N/D .