Review of group representations

1. Definitions

① (T, V)

$G \longrightarrow GL(V) \cong GL(n, k)$ with a basis $\{e_i\}$

$g \longmapsto T(g) \longmapsto M(g)$

$$T(g)\cdot e_j = \sum_i M_{ij}\, e_i$$

Key to construct a rep: How G-action is realized on a rep V.

V: "points", group algebra, $\rightarrow$ permutation

S_n on $\mathbb{R}^n = \mathrm{span}\{e_i\}$ $\quad \sigma\cdot e_i = e_{\sigma(i)}$

$\left\{ \begin{array}{l} \text{orbitals} \rightarrow \text{permu + phase} \\ \text{cosets of } H \rightarrow \text{permu + internal rep. of } H. \end{array} \right.$

if $H \triangleleft G$, then

$G \xrightarrow{\pi} G/H$ a natural homo.

$g \longmapsto gH$

$g_1 g_2 \overset{\pi}{\longmapsto} g_1 g_2 H \xrightarrow{\rho} \rho(g_1 g_2 H)$

$\rho\circ\pi$ a rep of G as well

$G \rightarrow G/G$ realizes trivial rep.

② (T_1, V_1) (T_2, V_2) reps of G.

intertwiner $A : V_1 \to V_2 \in \mathrm{Hom}_G(V_1, V_2)$

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1"'] & V_2 \arrow[d, "T_2"] \\ V_1 \arrow[r, "A"'] & V_2 \end{tikzcd}$$

$$T_2(g) A = A T_1(g)$$

if $T_2(g) = A T_1(g) A^{-1}$.

$T_1 \sim T_2$ equivalent reps.

③ character $\chi_T : G \to K$

$$\chi_T(g) := \mathrm{Tr}_V (T(g))$$

$T_1 \sim T_2$. then $\chi_{T_1}(g) = \chi_{T_2}(g)$

and $\chi(g_1) = \chi(g_2)$ if $g_1, g_2 \in$ same conj. class

We know ② → ③. but needs later orthogonality to know ③ → ②

④ $V_1 \oplus V_2$. $\chi_\oplus = \chi_1 + \chi_2$

$V_1 \otimes V_2$: $\chi_\otimes = \chi_1 \cdot \chi_2$

$\oplus$ helps to define decomposition later.

$\otimes$ often appears in physics

$V_1 \otimes V_2 \cong \oplus a_\mu V_\mu$ CG-coeff.

selection rules etc.

2. Unitary reps. U

$\langle U(g)u, U(g)v \rangle = \langle u, v \rangle$

To define inner product, first define

Haar measure:

$$\int_G d\mu(g)\, f(hg) = \int_G d\mu(g)\, f(g)$$

exists for locally compact groups; compact left = right.

finite $\frac{1}{|G|}\sum_g$

SU(2) $\frac{1}{16\pi^2}\sin\theta\, d\theta\, d\phi\, d\psi$ θ, ϕ, ψ domains;

unitarizable $T \sim U$.

define inner product $\langle u, v \rangle = \int_G d\mu(g) \langle T(g)u, T(g)v \rangle$

finite groups. $H = \sum_g \rho^\dagger \rho$ $\quad uHu^\dagger = 1$ then

$\hat{\rho} = H^{\frac{1}{2}} \rho H^{-\frac{1}{2}}$ is unitary

→ works for compact as well. but not nessory

in most cases.

We deal with either already unitarized.

or noncompact

3. regular rep

Action of $G \times G$ on G.

$$(g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1}$$

induced action on $f \in \mathrm{Map}(G \to \mathbb{C})$

$$[(g_1, g_2) \cdot f](g_0) = f(g_1^{-1} g_0 g_2)$$

Hilbert space $L^2(G) = \{ f : G \to \mathbb{C} \mid \int_G |f(g)|^2 d\mu(g) < \infty \}$

becomes a rep of $G \times G$.

and subgroups $G \times \{1\} \cong G$ left reg. rep

$\{1\} \times G \cong G$. right. reg. rep.

For finite groups. one can define the

"δ-basis". $\delta_g(h) = \begin{cases} 1 & g = h \\ 0 & \end{cases}$;

$g_1 \cdot \delta_{g_2}(h) = \delta_{g_2}(g_1^{-1} h) = 1$ iff $g_1^{-1} h = g_2$. $h = g_1 \cdot g_2$

i.e. $g_1 \delta_{g_2} = \delta_{g_1 g_2}$

This is the same as in the later

group algebra language. $v = \sum \alpha_g g \to |v\rangle = \sum \alpha_g |g\rangle$

Why important? $\to$ contains all irreps

Peter-Weyl.

4. reducible & irreducible;

W invariant under G: $\forall g \in G$. $\forall w \in W$.

$gw \in W$.

V has invariant nontrivial subspaces W? reducible

$V \cong \oplus W_i$. completely reducible

Examples. ① any rep of Abelian $\to \oplus$ 1D

② $\mathbb{R}^n$ as rep of S_n. $\mathbb{R}^n \cong W \oplus W^\perp$

$W = \mathrm{span}\{\sum e_i\}$

③ reg. rep. $\to$ all irreps

④ f.d. unitary ; compact.

<u>isotypic decomposition</u>

$$V \cong \oplus K^{a_\mu} \otimes V^\mu =: \oplus a_\mu V^\mu.$$

5. Schur's lemma

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1"'] & V_2 \arrow[d, "T_2"] \\ V_1 \arrow[r] & V_2 \end{tikzcd}$$

$$T_2(g)A = A T_1(g)$$

① $A = 0$ or isomorpism

② $V_1 = V_2 = V$. $A = \lambda \mathbb{1}_V$.

▷ Hamiltonians as intertwiners

$\Rightarrow$ block structure.

6. Using Schur's lemma, we show Peter-Weyl.

$$L^2(G) \cong \bigoplus_\mu \mathrm{End}(V^\mu)$$

isotypic decomposition over all distinct irreps

① Mat. element of unitary reps, orthogonal

$$\int_G \overline{\mathcal{U}^\mu_{ij}(g)}\, \mathcal{U}^\nu_{k\ell}(g)\, d\mu(g) = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ik}\delta_{j\ell}$$

② $\mathrm{Tr} \to \chi$

$$\int_G \overline{\chi^\mu(g)}\, \chi^\nu(g)\, d\mu(g) = \delta_{\mu\nu}$$

↳ character tables

	[E] $m_1[C_1]$	$m_2[C_2]$	$\cdots$
V_μ	χ_μ		
V_ν			
$\vdots$			

rows: $$\frac{1}{|G|}\sum_{C_i} m_i\, \overline{\chi_\mu([C_i])}\, \chi_\nu([C_i]) = \delta_{\mu\nu}$$

cols: $$\sum_\mu \overline{\chi_\mu([C_i])}\, \chi_\mu([C_j]) = \frac{|G|}{m_i}\delta_{ij}$$

not all characters are real. although in lectures we see real ones S_n, D_4, etc.

③ Any $V \cong \bigoplus a_\mu V^\mu$ then

$a_\mu = \langle \chi_\mu . \chi_V \rangle$

$V_1 \otimes V_2 \cong \bigoplus a_\mu V^\mu$

$a_\mu = \langle \chi_\mu . \chi_1 \cdot \chi_2 \rangle$

④ $|G| = \sum_\mu n_\mu^2$ for finite $\begin{cases} \text{a. matching dim} & \text{Peter-Weyl} \\ \text{b. } \chi_{r.r.}(e) = |G| . \ \chi(g \neq e) = 0 \end{cases}$

$$a_\mu = \langle \chi_{r.r} \ \chi_\mu \rangle = \frac{1}{|G|} |G| \cdot n_\mu = n_\mu$$

④ Projectors $P^\mu_{ij} = n_\mu \int_G \overline{\mathcal{U}^\mu_{ij}(g)} \, T(g) dg$

$$\mathrm{Tr} \to P^\mu = n_\mu \int_G \overline{\chi_\mu(g)} \, T(g) dg$$

$$P^\mu P^\nu = \delta_{\mu\nu} P^\mu .$$

trivial. $P^{triv} = \int_G T(g) \, dg$

7. Group ring & group algebra $\mathbb{C}[G]$

① class operators (CSCO)

↪ diagonalize → projectors <u>see notes</u>

$$\hat{C}_i \cdot \hat{C}_j = \sum_k [D_i]_{kj}\, \hat{C}_k$$

② S_n. use Young diagrams / tableaux

idempotent $c = \frac{n_\mu}{n!} PQ \quad P = \sum_{p \in R(T)} p \quad Q = \sum_{q \in C(T)} \epsilon(q) \cdot q$

n_μ = number of standard tableaux.

hook length formula $\frac{n!}{\prod h(b)}$

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \quad c = \frac{1}{2}(e - (12))$ etc.

8. Schur-Weyl duality $V = K^d$.

$$V^{\otimes n} \cong \bigoplus D^\lambda \otimes R^\lambda$$

irreps of $GL(d, K)$ and subgroups ↗ ↖ irreps of S_n

$SU(2)$. $V = \mathbb{C}^2 \quad R^\lambda \Leftrightarrow \underbrace{\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline \end{array}}_{T}$

$V_\ell = C(T) \cdot V^{\otimes n} \rightarrow$ irreps of dim $\ell + 1$

9. induced rep. $H \subset G$. rep (ρ, V)

$$\mathrm{Ind}_H^G(V) = \{ f : G \to V \mid f(gh^{-1}) = \rho(h) f(g) \}$$

H-equivariance

$$\mathrm{Ind}_H^G(V) \cong \bigoplus_{\underline{g_i H}} V^i . \qquad V_i \cong V$$

$$g_i H \xrightarrow{g} g_j H$$

$$g \cdot g_i = g_j \cdot h(g,i)$$

$$g \cdot v_a^{(i)} = \sum_{jb} \rho(h(g,i))_{ba} \, v_b^{(j)}$$

Define $f_{i.a}(g) = \begin{cases} \rho(h^{-1}) w_a & \text{if } g = g_i \cdot h \\ 0 & \end{cases}$

$$\text{the } [\underbrace{g \cdot \overbrace{f_{i.a}}^{\text{support on } g_i H}}_{\text{support on } g_j H}](g_j) = f_{ia}(g^{-1} g_j) = f_{ia}(g_i h^{-1})$$

$$= \rho(h) f_{ia}(g_i)$$

$$= \rho(h) f_{j.a}(g_j)$$

$$\chi_{\mathrm{Ind}}(g) = \sum_{g \cdot g_i H = g_i H} \chi_V [h(g,i)]$$

g does not change coset.

$\Rightarrow$ check the S_3 example.

$\Rightarrow SU(2)$ induced from ρ_{-k} of $U(1)$. and

restrict to holomorphic homogeneous polynomial

$\{ u^{j+m} v^{j-m} \}$

$\Rightarrow SO(3)$ by restricting to $\underline{\text{integer } j}$.

equiv. to Y_{lm}.

10. applications: point group operations:

on atomic orbitals (l integer)

$$\chi_l(R) = \frac{\sin(l+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} . \qquad R: \text{ proper rotation}$$

many point groups are just products with the inversion

$D_{6h} \cong D_6 \times C_i$ $\qquad C_i = \{e, i\}$

$\chi_l(R \otimes i) = (-1)^l \cdot \chi_l(R)$

character table is easily defined as

	$D_6 \otimes e$	$D_6 \otimes i$
g	χ	χ
u	χ	$-\chi$

in general. find the rep. space. determine character. reduce using character table.