Recap: Induced representation

① $\mathrm{Ind}_H^G(V) := \{ f: G \to V \mid f(gh^{-1}) = \rho(h) f(g) \quad \forall g \in G, \forall h \in H \}$

For $g \cdot g_i H = g_j \cdot H$ where g_i, g_j are representatives of cosets of H.

i.e. $g g_i = g_j h$

Then $(g \cdot f)(g_j) = f(g^{-1} g_j) = f(g_i h^{-1}) = \rho(h) f(g_i)$

The action of g on f:

① changes its support from $g_i H$ to $g_j H$.

② acts on it by $\rho(h)$

We can define $f_{i,a}(g_i) = w_a$ and

$$f_{i,a}(g) = \begin{cases} \rho(h^{-1}) w_a & \text{if } g = g_i \cdot h \\ 0 \end{cases}$$

② induce rep from $D = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \sim U(1) \subset SU(2)$

$\rho_k = e^{ik\theta}$ on $V = \mathbb{C}$, then

$\mathrm{Ind}_{U(1)}^{SU(2)}(\rho_{-k}) = \{ f: SU(2) \to \mathbb{C} \mid f(ue^{i\theta}, ve^{i\theta}) = f(u,v) \}$

$(u, v) := \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}$

8.16.1 homogeneous polynomials

+ holomorphic

$\Rightarrow u^{j+m} v^{j-m} =: f_{j,m} \qquad m=-j,\dots,j \qquad \dim = 2j+1$

If it really is a rep. then $\forall g \in SU(2)$

$$(g\cdot f_{j,m})(u,v) = f_{j,m}\left(\begin{pmatrix}\alpha & -\bar\beta \\ \beta & \bar\alpha\end{pmatrix}^{-1}\cdot\begin{pmatrix}u & -\bar v \\ v & \bar u\end{pmatrix}\right)$$

$$= f_{j,m}(\bar\alpha u + \bar\beta v, -\beta u + \alpha v)$$

$$= (\bar\alpha u + \bar\beta v)^{j+m}(-\beta u + \alpha v)^{j-m}$$

$$=: \sum_{m'} \tilde D^j_{m'm}(g) f_{j,m'}$$

Diagonal elements ($\beta=0$) acts as
$\bar\alpha = \alpha^{+}$

$$g\cdot f_{j,m} = \bar\alpha^{j+m}\alpha^{j-m} = \alpha^{-2m} f_{j,m}$$

i.e. $\hat D^j_{m'm} = \alpha^{-2m}\delta_{mm'}$

see similarity in $\underline{J_z}$. diagonal in $|j,m\rangle$ basis.

For general α,β.

$$(*) = \sum_{\substack{s,t\\(s,t\geq 0)}} \underbrace{\binom{j+m}{s}\binom{j-m}{t}\bar\alpha^{s}\alpha^{j-m-t}\bar\beta^{j+m-s}(-\beta)^{t}}\, u^{s+t} v^{2j-s-t}$$

$s+t \to j+m'$
$2j-s-t \to j-m'$

$$\tilde D^j_{m'm}(g) = \sum_{s+t=j+m'} \underline{\qquad} .$$

$$j=\tfrac12 \Rightarrow \tilde D^{\frac12}(g) = \begin{pmatrix}\bar\alpha & \bar\beta \\ -\beta & \alpha\end{pmatrix} = g^{+}$$

Remark. reps of $SO(3)$

recall the homomorphism $\pi : SU(2) \to SO(3)$

$$u\vec{x}\cdot\vec{\sigma}u^{-1} = (\pi(u)\vec{x})\cdot\vec{\sigma}$$

with $\pi(u) = \pi(-u)$

with the corresponding central extension

$$1 \to \mathbb{Z}_2 \xrightarrow{\iota} SU(2) \xrightarrow{\pi} SO(3) \to 1$$

i.e. $\boxed{SO(3) \cong SU(2)/\mathbb{Z}_2}$

We can then obtain the irreps of $SO(3)$

Note that $\tilde{D}^j_{m'm} = \delta_{m'm}\,\alpha^{-2m}$ for diagonal matrices. so $g = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ acts on V_j as

$$(-1)^{-2m}\,\mathbb{1}_{V_j}$$

which should act trivially for all m.

then m has to be integer. so does j.

($\tilde{D}^j \to SO(3)$ $\ker = \mathbb{1}$ iff $j =$ integer)

So the irreps of $SO(3)$ are given by

V_j with $j \in \mathbb{Z}$. and thus $\dim_{\mathbb{C}} V_j = 2j+1$ odd.

8.16.2 Characters and irreducibility

We actually know from Schur-Weyl that V_j are irreps

① This also means the characters are ON. i.e.

$$\langle \chi_j , \chi_{j'} \rangle = \delta$$

$g \sim d(z) = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$ $z = e^{i\theta}$ defines a conjugacy class.

$$\tilde{D}^j_{m'm}(d(z)) = \sum_{\substack{s+t=j+m' \\ \left\{ \begin{subarray}{l} t=0 \\ j+m=s \end{subarray} \right.}} \bar{\alpha}^{j+m} \alpha^{j-m} = z^{-2m} \delta_{mm'}$$

$$\tilde{D}^j(d(z)) = \mathrm{diag}\{ z^{-2j}, z^{-2j+2}, \cdots, z^{2j} \}$$

$$\chi_j(g) = z^{-2j}(1 + z^2 + \cdots + z^{4j}) = \frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}} = \frac{\sin(2j+1)\theta}{\sin\theta}$$

Chebyshev polynom. of 2nd kind.

To compute the inner product, we need Haar measure.

Earlier. we parametrize as

$$g = e^{i\frac{1}{2}\phi\sigma^3} e^{i\frac{1}{2}\theta\sigma^1} e^{i\frac{1}{2}\psi\sigma^3} \quad \text{with}$$

$$d\mu_g = \frac{1}{16\pi^2} d\phi\, d\psi \,\underline{\sin\theta\, d\theta} \qquad \begin{array}{l} \phi \in [0, 2\pi) \\ \psi \in [0, 4\pi) \\ \theta \in [0, \pi) \end{array}$$

$$\mathrm{Tr}\, g = \cos\frac{\theta}{2} \cos\frac{\phi+\psi}{2}.$$

change variable to the "θ" here for the class function

We get Haar measure $dg = \frac{2}{\pi} \sin^2\theta\, d\theta$

Alternatively. we see that $g = \cos\theta\, \mathbb{1} + i\sin\theta\,(\hat{n}\cdot\hat{\sigma})$ $\theta \in [0,\pi]$, $\hat{n} \in S^2$

each θ labels a conj. class. $\mathrm{tr}\, g = 2\cos\theta$

$$C_\theta = \{ g(\theta, \hat{n}) \mid \hat{n} \in S^2 \}$$

Fix θ. $\hat{n}$ runs over a 2-sphere with radius $\sin\theta$.

and $S^3 = \bigcup_{\theta\in[0,\pi]} C_\theta$

So the volume of S^3 is just volume of C_θ integrated over θ. $\propto \int \sin^2\theta\, \underbrace{d\Omega}_{4\pi}\, d\theta$

$$\frac{2}{\pi}\int_0^{\pi} f(\theta)\sin^2\theta\, d\theta = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin^2\theta\, d\theta = -\frac{1}{4\pi i}\oint f(z)\,(z - z^{-1})^2\,\frac{dz}{z}$$

$(f(\theta) = \overline{\chi_j}\,\chi_{j'}$. even in $\theta)$

$$= \frac{1}{4\pi i}\oint f(z)\,\overline{(z - z^{-1})}(z - z^{-1})\,\frac{dz}{z}$$

$$\langle \chi_j, \chi_{j'} \rangle = \frac{1}{4\pi i}\oint \overline{(z^{2j+1} - z^{-2j-1})}\,(z^{2j'+1} - z^{-2j'-1})\,\frac{dz}{z}$$

$\overline{z} = z^{-1}$

$\ell = 2j+1$

$$= \frac{1}{4\pi i}\oint \Big(\underbrace{z^{\ell'-\ell-1}}_{2\pi i\,\delta_{\ell\ell'}} - z^{-\ell-\ell'-1} - z^{\ell+\ell'-1} + \underbrace{z^{\ell-\ell'-1}}_{2\pi i\,\delta_{\ell\ell'}}\Big)\, dz$$

$$= \delta_{jj'}$$

② check the self-intertwiner A.

$$A\hat{D} - \tilde{D}A = 0$$

$$\sum_n A_{mn}\,\hat{D}^j_{n\ell} = \sum_n \tilde{D}^j_{mn}\,A_{n\ell}$$

$$\sum_n A_{mn} z^{-2n} \delta_{nl} = \sum_n z^{-2m} \delta_{mn} A_{nl}$$

$$A_{ml} z^{-2l} = z^{-2m} A_{ml}$$

$$A_{ml}(z^{-2l} - z^{-2m}) = 0 \qquad \forall z.$$

$$\Rightarrow A_{ml} = a_m \delta_{ml}.$$

For arbitrary $\tilde{D}$. $(A\tilde{D})_{ml} = (\tilde{D}A)_{ml}$ $\Rightarrow A_{mm} = A_{ll}$.

$$A_{mm} \tilde{D}_{ml} = \tilde{D}_{ml} A_{ll}$$

$\Rightarrow A = a \cdot \mathbb{1}_{2j}$ is the only possible self-intertwiner.

Shur's lemma $\Rightarrow$ irrep.

8.16.3 Unitarization

$\tilde{f}^m_{2j} = u^{j+m} v^{j-m}$, $u, v \in \mathbb{C}^2$ $\quad \bar{f} f$ diverges

$$\langle f_1, f_2 \rangle_{H_{2j}} = \frac{1}{\pi (2j+1)!} \int_{\mathbb{C}^2} \overline{f_1(u,v)} f_2(u,v) e^{-(|u|^2+|v|^2)} d^2u \, d^2v$$

$$\langle g f_1, g f_2 \rangle_{V_j} \overset{g \text{ unitary}}{=} \langle f_1, f_2 \rangle_{V_j}$$

$$\hookrightarrow \quad f_{j,m} = \frac{1}{\sqrt{\pi}} \sqrt{\frac{(2j+1)!}{(j+m)!(j-m)!}} u^{j+m} v^{j-m}$$

$$\tilde{D}^j_{m'm}(g) = \sum_{s+t=j+m'} \binom{j+m}{s} \binom{j-m}{t} \bar{\alpha}^s \alpha^{j-m-t} \bar{\beta}^{j+m-s} (-\beta)^t$$

take earlier parametrization:

$$\alpha = e^{i\frac{1}{2}(\psi+\phi)}\cos\frac{\theta}{2} \qquad \beta = -e^{i\frac{1}{2}(\psi-\phi)}\sin\frac{\theta}{2}$$

$$\tilde{D}_{m'm}(g) = \sum_t \binom{j+m}{j+m'-t}\binom{j-m}{t} e^{i\frac{1}{2}(\psi+\phi)[j-m-t-(j+m'-t)]}$$

$s = j+m'-t$

$$(-1)^{j+m-(j+m'-t)} \times e^{i\frac{1}{2}(\psi-\phi)[t-j-m+j+m'-t]}$$

$$\left(\cos\frac{\theta}{2}\right)^{j-m-t+(j+m'-t)}\left(\sin\frac{\theta}{2}\right)^{t+j+m-(j+m'-t)}$$

$$= (j+m)!\,(j-m)!\, e^{-i(m\psi+m'\phi)} \times \sum_t \left[(-1)^{t+m-m'} \frac{\left(\cos\frac{\theta}{2}\right)^{2j-m-m'-2t}\left(\sin\frac{\theta}{2}\right)^{m-m'+2t}}{(j+m'-t)!\,(m-m'+t)!\,(j-m-t)!\,t!} \right]$$

$$= \underbrace{e^{-im'\phi}\, d^j_{m'm}(\theta)\, e^{-im\psi}}_{\text{Wigner D-matrix}} \sqrt{\frac{(j+m)!\,(j-m)!}{(j+m')!\,(j-m')!}}$$

$$d^j_{m'm}(\theta) = [(j+m')!\,(j-m')!\,(j+m)!\,(j-m)!]^{\frac{1}{2}} \sum_t [\;\cdots\;]$$

$\tilde{D}$ is related to the Wigner-D matrix in physics

$$D^j_{m'm} = \langle j, m' |\exp\left(\frac{-\hat{J}\cdot\hat{n}\,\phi}{\hbar}\right)| j, m\rangle$$

as $$D^j_{m'm} = \sqrt{\frac{(j+m')!\,(j-m')!}{(j+m)!\,(j-m)!}}\;\tilde{D}^j_{m'm}$$

D unitary in $|j,m\rangle$ basis

another evidence that we can identify $\underline{u^{j+m} v^{j-m} \Leftrightarrow |j,m\rangle}$

(easily seen for integer spins. $u^{j+m} v^{j-m} \propto Y_j^m(\theta,\phi)$)

8.16.4 The Clebsch-Gordan decomposition of SU(2)

(§11.20 Moore) (Tinkham. GT & QM. book)

Now consider $V_{j_1} \otimes V_{j_2}$. decompose using character theory:

$$\chi_j(z) = \frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}}$$

$j_1 = \frac{1}{2}$. then $\chi_{1/2} = z + z^{-1}$

$$\chi_{\frac{1}{2}\otimes j} = \chi_{1/2}\chi_j = (z + z^{-1}) \frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}}$$

$$= \frac{(z^{2j+2} - z^{-2j-2}) + (z^{2j} - z^{-2j})}{z - z^{-1}}$$

$$= \chi_{j+\frac{1}{2}} + \chi_{j-\frac{1}{2}}$$

$$\Rightarrow V_{\frac{1}{2}} \otimes V_j \cong V_{j+\frac{1}{2}} \oplus V_{j-\frac{1}{2}}$$

in general

$$\chi_{j_1\otimes j_2} = \chi_{j_1}\cdot\chi_{j_2} = \frac{z^{2j_1+1} - z^{-2j_1-1}}{z - z^{-1}} \cdot \frac{z^{2j_2+1} - z^{-2j_2-1}}{z - z^{-1}}$$

$$= \sum_{j=j_1-j_2}^{j_1+j_2} \frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}} \quad (j_1 \geq j_2)$$

$$= \sum_{j=j_1-j_2}^{j_1+j_2} \chi_j$$

or equivalently

$$\langle \chi_{j_1}\chi_{j_2}, \chi_j \rangle = \begin{cases} 1. & j_1 - j_2 \leq j \leq j_1 + j_2 \\ 0 \end{cases} \quad \left(\begin{array}{l} \langle \chi_j, \chi_{j'} \rangle \\ = \delta_{jj'} \end{array} \right)$$

$$\Rightarrow V_{j_1} \otimes V_{j_2} \cong \bigoplus_{j=j_1-j_2}^{j_1+j_2} V_j$$

Clebsch-Gordan coefficient:

$\{\psi_{j,m}\}$ an orthonormal basis set of V_j

P_j a projector from $V_{j_1} \otimes V_{j_2}$ onto V_j

$\langle jm \,,\, P_j(\psi_{j_1 m_1} \otimes \psi_{j_2 m_2}) \rangle$ — CG. coefficient.

$\langle jm \mid j_1 m_1 ; j_2 m_2 \rangle$ in physics, often expressed as "Wigner-3j"

symbols: $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1 \, j_2 m_2 \mid j_3 (-m_3) \rangle$

$\left(\begin{array}{l} m_1 + m_2 + m_3 = 0 : \text{ conservation of } m \\ j_i + j_j \geq j_k : \text{ triangular relation} \end{array} \right)$

$$|jm\rangle = \sum_{m_1, m_2} |j_1 m_1 ; j_2 m_2\rangle \underbrace{\langle j_1 m_1 ; j_2 m_2 | jm\rangle}_{CG - \text{coeff.}}$$

trivial rep: $P = \int_G T(g)\, dg \qquad \alpha, \beta \in \{+,-\}$

$$T(g) |\alpha\rangle \otimes |\beta\rangle = \sum_{\gamma, \delta} g_{\gamma\alpha}\, g_{\delta\beta}\, |\gamma\rangle \otimes |\delta\rangle$$

HW07: $\int_{SU(2)} dg \; g_{\alpha\beta}\, g_{\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} \Rightarrow P = \frac{1}{2} \epsilon_{\gamma\delta} \epsilon_{\alpha\beta}$

Thus $P|\alpha\rangle \otimes |\beta\rangle = \sum_{\gamma, \delta} \frac{1}{2} \epsilon_{\gamma\delta} \epsilon_{\alpha\beta} |\gamma\rangle \otimes |\delta\rangle$

$= \frac{1}{2} \epsilon_{\alpha\beta} \left(|+,-\rangle - |-,+\rangle \right)$

$\psi_s = \frac{1}{\sqrt{2}} \left(|+\rangle|-\rangle - |-\rangle|+\rangle \right)$

CG: $\langle 0,0 \mid \frac{1}{2}, \pm\frac{1}{2} ; \frac{1}{2}, \mp\frac{1}{2} \rangle = \pm\frac{1}{\sqrt{2}}$ otherwise 0

Connection to Wigner D-matrices: $g \in SU(2) / SO(3)$

$$g \cdot \psi_{j_1 m_1} = \sum_{m_1'} D^{j_1}_{m_1' m_1}(g) \psi_{j_1 m_1'}$$

ψ_{jm}. ON basis of irrep j.

$$g \cdot \psi_{j_2 m_2} = \sum_{m_2'} D^{j_2}_{m_2' m_2}(g) \psi_{j_2 m_2'}$$

recall $g(\psi_{j_1 m_1} \otimes \psi_{j_2 m_2}) = g\psi_{j_1 m_1} \otimes g\psi_{j_2 m_2}$

$$= \sum_{m_1' m_2'} \underbrace{D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2}}_{(D^{j_1} \otimes D^{j_2})_{m_1' m_2', m_1 m_2}} \psi_{j_1 m_1'} \psi_{j_2 m_2'}$$

$D^{j_1} \otimes D^{j_2} \simeq \bigoplus_{|j_1 - j_2|}^{j_1 + j_2} D^{j} \rightarrow$ labeled in JM.

$$= A^{-1} M(g) A \qquad M(g) = \delta_{jj'} D_{m'm}$$

$$D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2} = \sum_{j, m, m'} A^{-1}{}_{m_1' m_2', j m'} D^{j}_{m' m} A_{jm, m_1 m_2}$$

$$\psi^{j}_{m} = \sum_{m_1 m_2} \psi^{j_1}_{m_1} \psi^{j_2}_{m_2} (A^{-1})_{m_1 m_2 j m} \quad \text{or}$$

$$\psi^{j_1}_{m_1} \psi^{j_2}_{m_2} = \sum_{JM} \psi^{j}_{m} A_{jm, m_1 m_2}$$

A transforming between two ON basis $\Rightarrow$ unitary

$$A_{JM, m_1 m_2} = \langle \psi^{j}_{m} | \psi^{j_1}_{m_1} \psi^{j_2}_{m_2} \rangle \quad \text{CG-coefficients}$$

$$D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2} = \sum_{|j_1 - j_2|}^{j_1 + j_2} \sum_{m m'} \langle j m' | j_1 m_1', j_2 m_2' \rangle \langle j m | j_1 m_1, j_2 m_2 \rangle D^{j}_{m' m}$$

Skip details here. only mention the last line

8.16.5 Wigner-Eckart theorem.

For systems with rotational symmetry the states transforming following irreps $\underline{j}$

ψ^{α}_{jm}. where α labels other "quantum numbers", and m indices withing irrep j.

➡ How does an operator look like within an irrep?

group action on operators

After rotation. $\hat{O} \to \hat{O}'$. $\psi \to \psi'$, then

$$\hat{O}'\psi' = (\hat{O}\psi)'$$
$$\hat{O}'(g\psi) = g(\hat{O}\psi) = g\hat{O}g^{-1}(g\psi)$$
$$\Rightarrow \hat{O}' = g\hat{O}g^{-1}$$

For observables carrying an angular momentum. they can be expressed in the basis of Irreducible Tensor operators,

(operators transforming as irreps of rotational group.)

$$g\, O^{j}_{m}\, g^{-1} = \sum_{m'} D^{j}_{m'm}\, O^{j}_{m'} \qquad m, m' = -j, \cdots j$$

examples: rank-0: total energy $(\hat{H})$ density $(\hat{N})$

rank-1: angular momentum J

dipole operator $\vec{r}$

$$\langle \psi^{\alpha}_{j_1 m_1} | \hat{O}^{j}_{m} | \psi^{\beta}_{j_2 m_2} \rangle = \langle \psi^{\alpha}_{j_1 m_1} | g^{-1} g \underbrace{\hat{O}^{j}_{m} g^{-1}} g | \psi^{\beta}_{j_2 m_2} \rangle$$

$$= \langle \sum_{m_1'} \psi^{\alpha}_{j_1 m_1'} D^{j_1}_{m_1' m_1} | (\sum_{m'} D^{j}_{m' m} \hat{O}^{j}_{m}) | \sum_{m_2'} \psi^{\beta}_{j_2 m_2'} D^{j}_{m_2' m_2} \rangle$$

$$= \sum_{m_1', m_2', m'} \overline{D^{j_1}_{m_1' m_1}} \underbrace{D^{j}_{m' m} D^{j_2}_{m_2' m_2}} \langle \psi^{\alpha}_{j_1 m_1'} | \hat{O}^{j}_{m} | \psi^{\beta}_{j_2 m_2'} \rangle$$

insert $D^{j}_{m' m} D^{j_2}_{m_2' m_2} = \sum_{|j-j_2|}^{j+j_2} \sum_{m_3 m_3'} \langle j_3 m_3' | j\, m'\, j_2 m_2' \rangle \langle j_3 m_3 | j\, m\, j_2 m_2 \rangle D^{j}_{m_3' m_3}$

from above, and use the orthonormal relation

"angular part"

$$\langle \alpha j_1 m_1 | \hat{O}^{j}_{m} | \beta j_2 m_2 \rangle = \langle j_1 m_1 | j\, m ; j_2 m_2 \rangle \times$$

$$\underbrace{\left(\sum_{m_1' m_2' m'} \langle j_1 m_1' | j\, m' ; j_2 m_2' \rangle \langle \alpha j_1 m_1' | \hat{O}^{j}_{m'} | \beta j_2 m_2' \rangle \right)}_{\langle j_1 \| \hat{O}^{j} \| j_2 \rangle}$$

reduced matrix element.

independent of m's.

"radial part"

The above is the statement of the Wigner–Eckert theorem:

Matrix element of a tensor-operator factorizes into a CG coefficient + a reduced mat. element. independent of magnetic Q.N.

=> selection rules from the angular part.

often in atomic/spectroscopic contexts.

9. Crystalline point groups

Ref: ① Dreselhaus. Chap 5 – 13 ;
② Bradley & Cracknell.
"The mathematical theory of symmetry in solids" ;

In E^3. $\vec{r}' = R(g)\cdot\vec{r}$ $\quad g\cdot\hat{e}_j = \sum_i R(g)_{ij}\,\hat{e}_i$

length/angle invariant after symmetry operation

$\langle \vec{r}', \vec{s}' \rangle = \vec{r}'^T\cdot\vec{s}' = \vec{r}^T\cdot R(g)^T\cdot R(g)\cdot\vec{s} \equiv \vec{r}^T\cdot\vec{s} \quad (\forall \vec{r}, \vec{s})$

$\Rightarrow R(g)^T\cdot R(g) = \mathbb{1} \qquad R\in O(3, \mathbb{R})$

$R\in SO(3)$ $\det = 1$ proper rotation

$R\in PSO(3)$ $\det = -1$ improper rotation:

prop. rot. ∘ inversion

<u>Point groups</u>: subgroups of rotation group $O(3)$.

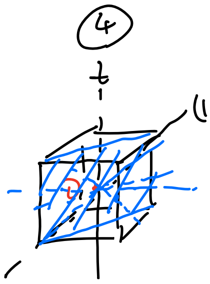
9.1. Symmetry operations (Dresselhaus 3.9)

- E identity
- C_n rotations of $\frac{2\pi}{n}$

$C_2 : \pi \qquad C_3 : \frac{2\pi}{3} \qquad C_3^2 : \frac{4\pi}{3}$ etc.

in crystalline systems: $n = 1, 2, 3, 4$ or 6

- σ : reflection in a plane



σ_h horizontal perp to } principal rot. axis (highest order)

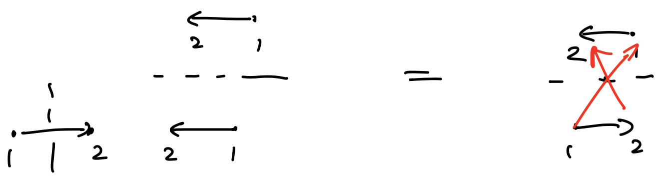
σ_v vertical contains

σ_d "dihedral plane" bisects the angle between 2-fold axes perp. to principal axis

- i : inversion $\vec{r} \to -\vec{r}$

$$i = \sigma C_2 (= S_2)$$

$$S_n := \sigma C_n$$



- S_n : $2\pi/n$ rotation $+ \sigma_h$

n odd. $iC_n^k = \sigma C_2 C_n^k = \sigma (C_{2n}^n C_{2n}^{2k}) = \sigma C_{2n}^{n+2k}$

$= S_{2n}^{n+2k}$

n even: $iC_n^k = \sigma C_2 C_n^k = \sigma C_n^{\frac{n}{2}+k}$

$iC_3^{\pm} = S_6^5 = S_6^{\mp}$

$iC_4^{\pm} = S_4^{\mp}$

$iC_6^{\pm} = S_3^{\mp} \, (= \sigma C_6^{3\pm 1} = \sigma C_3^{\mp})$

Above are Schönflies notations

often see Hermann-Mauguin notations

("international notations")

Schönflies	H-M
C_n	n
iC_n	$\bar{n}$
σ	m
σ_h	n/m
σ_v	nm
$\sigma_{v'}$	nmm

Examples.

	proper		improper	
	S	HM	S	HM
	$C_1 = E$	1	$i = S_2$	$\bar{1}$
	C_2	2	$iC_2 = \sigma$	$\bar{2}$
	C_3^+	3	S_6^-	$\bar{3}$
	$C_3^2 = C_3^-$	3_2	S_6	$\bar{3}_2$
	C_4	4	S_4^-	$\bar{4}$
	C_4^-	4_3	S_4	$\bar{4}_3$
	C_6	6	S_3^-	$\bar{6}$
	C_6^-	6_5	S_3	$\bar{6}_5$

9.2 Point groups

Proper point groups

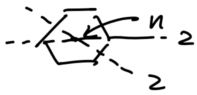
(P.G. of the first kind)

① cyclic groups. sym. elements. only n-fold rot.

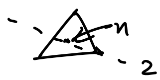
n (C_n)

② dihedral : n-sided prism

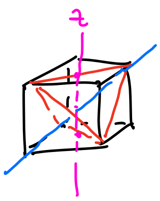
(D_n) { even : $n22$



odd : $n2$



③ tetrahedral regular tetrahedron.



23 (T)

3 2-fold // x.y.z . 4 3-fold .

12 : $E + C_{3j}^{\pm} + C_{2x/y/z} = 12$

$1 + 8 + 3$

For these. [001] first then [111] then [110]

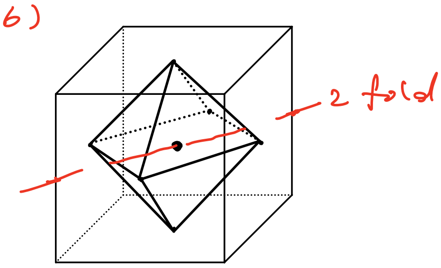
④ octahedral group

432 (O)

2-fold (6)

4-fold (x.y.z)

3-fold (4)



$E + C_{4xyz}^{\pm} + C_{2xyz} + C_{3j}^{\pm} + C_{2p} = 24$

$1 \quad 6 \quad 3 \quad 8 \quad 6$