Recap: Schur-Weyl duality

$V = K^d$. rep of $GL(d,K)$ (seen as rep of G.)

$$V^{\otimes n} \cong \bigoplus D_\lambda \otimes R_\lambda$$

irrep of GL and its subgroups

irrep of S_n

depends on the observation that the action of G and S_n commutes on $V^{\otimes n}$

Nothing special about $SU(2)$, which we used as an example.

In fact, if we use $SO(3)$ rep on $V = \mathbb{R}^3$ we can similarly work out the irreps

$$V^{\otimes 2} \cong Sym^3 V \oplus \Lambda^3 V.$$

$Sym^3 V = \text{span}\{ e_i \otimes e_j + e_j \otimes e_i,\ i,j \in \{1,2,3\}\}$

$\dim = \frac{d(d+1)}{2} = 6$

$\Lambda^3 V = \text{span}\{ e_i \otimes e_j - e_j \otimes e_i \}$

$\dim = \frac{d(d-1)}{2} = 3$

$xx \equiv e_1 \otimes e_1$, etc. then

$Sym^2 V \cong \{ xx, yy, zz, \frac{xy+yx}{2}, \frac{xz+zx}{2}, \frac{yz+zy}{2} \}$

$\Lambda^2 V \cong \{ \frac{xy-yx}{2}, \frac{xz-zx}{2}, \frac{yz-zy}{2} \}$

8.16. Induced representation

Refs: ① Moore . § 11.18 (11.18.1 not necessary)

② 陶瑞宝

8.16.1 Introduction.

Often we know how states transform under a subgroup $H \subset G$. (i.e. we have a representation of H). We want to know how it transforms under the full group (i.e. rep of G)

point groups ⊂ space groups
little groups

$\Rightarrow$ rep of G induced from rep of H.

For a sub group H. we have cosets

$$G/H = \{ g_1 H, \cdots g_r H \} = \bigcup_{i=1}^{r} g_i H$$

Let us see each coset as a vector, then span $\{g_i H\}$ ($\dim = r$) is invariant under left group action: $\forall g . g_i$ $\exists g_j \in G$. s.t.

$$g \cdot (g_i H) = g_j H$$

i.e. span $\{g_i H\}$ carries a rep of G.

For any function defined on the cosets G/H.

$$f : G/H \to \mathbb{C}$$

we define induced action as usual.

$$(g\cdot f)(g_i H) = f(g^{-1}\cdot g_i H) \ .$$

Now. we know the rep of H on some vector space:

$$\rho : H \longrightarrow GL(V)$$
$$h \mapsto \rho(h)$$

Then we can attach the same V to each coset $g_i H$. which form a vector space

$$W \cong \bigoplus_{i=1}^{r} V_i \qquad V_i \cong V := \text{span}\{v_1, \dots v_d\}$$

a vector of W is given as $\{v^{(1)}_{\alpha_1} \ . \ v^{(2)}_{\alpha_2}, \ \cdots \ v^{(r)}_{\alpha_r}\}$

How does G acts on this space?

Suppose $\forall g \in G$. $g\, g_i H = g_j H$. then

$$g\, g_i = g_{j(g,i)} h(g,i), \quad h(g,i) \in H$$

We can define g action on $w = (v^{(1)}, v^{(2)}, \dots v^{(r)}) \in W$ as

1. pick out $v^{(i)} \in V_i$, move it to V_j

$$i \longmapsto j(g,i)$$

2. acts on it via $\rho(h)$:

$$g\cdot v^{(i)}_{\alpha} = \sum_{j\beta} [\rho(h(g,i))]_{\beta\alpha}\, v^{j(g,i)}_{\beta}$$

Then W carries a rep of G.

Now we develop this idea formally using function space.

8.16.2 Induced representation.

Now we define the induced representation as a vector space spanned by functions $f: G \to \mathbb{C}$ that are H-equivariant.

$$\mathrm{Ind}_H^G(V) := \{ f : G \to V \mid f(gh^{-1}) = \rho(h) f(g), \quad \forall g \in G,\ h \in H \}$$

By H-equivariant we mean

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "R(h)"'] & V \arrow[d, "\rho(h^{-1})"] \\ G \arrow[r, "f"'] & V \end{tikzcd}$$

right action

i.e. $f(g \cdot h) = \rho(h^{-1}) f(g)$

Why? Suppose $f(g_i) = v \in V$. then with the same coset relation as above

$$g g_i = g_{j(g,i)} h(g,i), \quad h(g,i) \in H$$

We have

$$(g \cdot f)(g_j) = f(g^{-1} g_j) = f(g_i h^{-1}) = \rho(h) f(g_i)$$

i.e. the new functin $(g \cdot f)$ takes a new value $\rho(h) \cdot v$

This is exactly the block construction above. where

$$g \cdot v_\alpha^{(i)} = \sum_{j\beta} [\rho(h(g,i))]_{\beta\alpha} v_\beta^{j(g,i)}$$

Here, in this function construction. Different copies V_i. V_j .. are realized by different support. f belonging to different "blocks" is defined by $f_i(g_j H) = 0$ if $i \neq j$.

We can define a basis for $\mathrm{Ind}_H^G V$:

$$f_{i,a}(g) = \begin{cases} \rho(h^{-1}) w_a & \text{if } g = g_i h \\ 0 & \text{otherwise} \end{cases}$$

$a = 1, \cdots \dim V$

if $g \cdot g_i = g_j h(g,i)$

$$(g \cdot f_{i,a})(g_j) = f_{i,a}(g^{-1} g_j) = f_{i,a}(g_i h(g,i)^{-1})$$
$$= \rho(h(g,i)) w_a = \rho(h(g,i)) f_{j,a}(g_j)$$

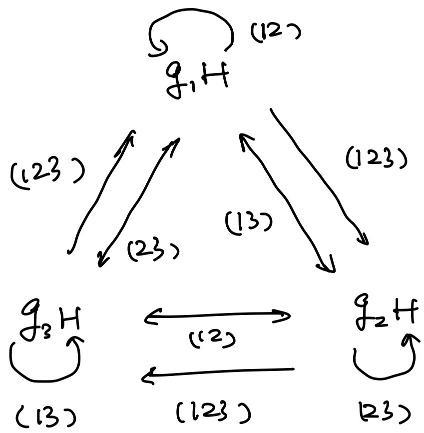
The character $\chi_{Ind}(g) = \sum_{\substack{g_i :\, g g_i H = g_i H \\ g g_i = g_i h}} \chi_V[h(g,i)] = \sum_{\substack{g_i : \\ g g_i H = g_i H}} \chi_V(g_i^{-1} g g_i)$

Example: $G = S_3$, $H = S_2 = \{e, (12)\}$

$$G/H = \{e, (12)\} \cup \{(13), (123)\} \cup \{(23), (132)\}$$

$$g_1 = e, \qquad g_2 = (13) \qquad g_3 = (23)$$

① Then S_3 acts on G/H as:



② Rep of $H = S_2$ on $\mathbb{C}$: $\rho(e) = 1$, $\rho(12) = \epsilon \in \{\pm 1\}$

③ Define basis that satisfy $f(gh^{-1}) = \rho_{\mathbb{C}}(h) f(g)$

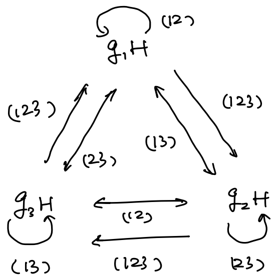
on g_1H: $f_1(e) = f_1((12)(12)) = \rho(12) f_1(12)$

g_2H: $f_2(123) = f_2[(13)(12)] = \rho(12) f_2(13)$

g_3H: $f_3(132) = f_3[(23)(12)] = \rho(12) f_3(23)$

We can thus choose

$$f_1(12) = f_2(13) = f_3(23) = 1$$



$g_1 = e$
$g_2 = (13)$
$g_3 = (23)$

④ We now construct the representation

$[(123) f_1](g) = f_1[(132) g]$

$(123) \cdot g_1H \to g_2H$. thus

$(123) f_1$ has support on g_2H.

Thus $[(123) f_1](g_2) = f_1((132) g_2)$

$= f_1((12)) = 1 = f_2(g_2)$

i.e. $(123) f_1 = f_2$

Similarly

$[(123) f_2](g_3) = f_2((132)(23)) = f_2(13) = 1 = f_3(g_3)$

$\Rightarrow (123) f_2 = f_3$

$[(123) f_3](g_1) = f_3((132)) = \epsilon = f_1(g_1)$

$\Rightarrow (123) f_3 = f_1$

$$\Rightarrow \rho_{Ind}((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\chi_{Ind}((123)) = \underbrace{\sum_{g_i : (123) g_i H = g_i H}}_{\text{no such case}} \chi_v(h) = 0$$

One can also work out $(12) f_1 = \epsilon f_1$, $(12) f_2 = \epsilon f_3$, $(12) f_3 = \epsilon f_2$

$$\Rightarrow \rho_{Ind}((12)) = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix}$$

$$\chi_{Ind}((12)) = \sum_{g_i : (12) g_i H = g_i H} \chi_v(h) = \chi_v((12)) = \epsilon$$

$(12) \cdot \underset{g_i}{\underline{e}} = \underset{g_j}{\underline{e}} \cdot \underset{h}{\underline{(12)}}$

or $\sum_{g_i} \chi_v(g_i^{-1} g g_i) = \chi_v((12))$

isotypic decomposition:

S_3	e	$3[(12)]$	$2[(123)]$
$(\epsilon=1)$ triv	1	1	1
$(\epsilon=-1)$ sgn	1	-1	1
std	2	0	-1
χ_{Ind}	3	ϵ	0

$$a_\epsilon = \frac{1}{6}(3 + 3\epsilon^2) = 1$$

$$a_{std} = \frac{1}{6}(6) = 1$$

$$\mathrm{Ind}_{S_2}^{S_3}(V(\epsilon)) \cong V(\epsilon) \oplus V_{std}$$

8.16. Representations of $SU(2)$.

Ref. Moore § 11.19

An element $g \in SU(2)$ is given as

$$g = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \qquad |u|^2 + |v|^2 = 1. \quad u, v \in \mathbb{C}.$$

it is clear that its diagonal subgroup

$$D = \left\{ \begin{pmatrix} z & \\ & \bar{z} = z^{-1} \end{pmatrix} \middle| \ |z| = 1 \right\} \cong U(1)$$

It has 1D irreps $\rho_k(z) = z^k$. Now we try to induce from it the reps of $SU(2)$:

$$\mathrm{Ind}_H^G(V) := \{ f : G \to V \mid f(gh^{-1}) = \rho(h) f(g), \ \forall g \in G, h \in H \}$$

$$\mathrm{Ind}_{U(1)}^{SU(2)}(\rho_{-k}) = \{ f : SU(2) \to \mathbb{C} \mid f(g \cdot \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}) = e^{ik\theta} f(g) \}$$

$$f\left[\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}\right] = f\left[\begin{pmatrix} ue^{i\theta} & -\bar{v}e^{-i\theta} \\ ve^{i\theta} & \bar{u}e^{-i\theta} \end{pmatrix}\right] := f(ue^{i\theta}, ve^{i\theta})$$

$$\Rightarrow f(ue^{i\theta}, ve^{i\theta}) = e^{ik\theta} f(u, v)$$

This form tells us that f has to be homogeneous polynomials of degree k.

Now. if we allow forms such as $u^{p_1} v^{p_2} \bar{u}^{q_1} \bar{v}^{q_2}$.

then, after $U(1)$ action, it become $e^{i\theta(p_1+p_2-q_1-q_2)} u^{p_1} v^{p_2} \bar{u}^{q_1} \bar{v}^{q_2}$

we require $p_1+p_2-q_1-q_2=k$. infinitely many choices. We require holomorphicity, i.e.

$$f = u+iv. \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or $\frac{\partial f}{\partial \bar{z}} = 0$. then $p_1+p_2=k$

$(k+1)$-dimensional.

8.16.1. Homogeneous polynomials

For $\{f_i\}$ to be a rep of $SU(2)$:

$$g \cdot f_i = \sum_j D(g)_{ji} f_j$$

We take $\{f_i\}$ to be homogeneous polynomials in u, v of degree $k=2j$ ($u^{j+m} v^{j-m}$, $m=-j, \cdots j$)

$\dim V_j = 2j+1$. $\Leftarrow$ this is the same as $Sym^{2j}(\mathbb{C}^2)$ we've seen earlier. and Schur-Weyl tells us that they are irreps

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \quad g^{-1} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix}$$

$$\begin{aligned} (g \cdot \tilde{f}_{j,m})(u,v) &= \hat{f}_{j,m}(\bar{\alpha} u + \bar{\beta} v, -\beta u + \alpha v) \\ &= (\bar{\alpha} u + \bar{\beta} v)^{j+m} (-\beta u + \alpha v)^{j-m} \quad (*) \\ &:= \sum_{m'} D^j_{m'm}(g) \hat{f}_{j,m'} \end{aligned}$$

$g \in U(1)$, $\beta = 0$: $g \cdot \tilde{f}_{j,m} = \bar{\alpha}^{j+m} \alpha^{j-m} \tilde{f}_{j,m} = \alpha^{-2m} \tilde{f}_{j,m}$

$\bar{\alpha} = \alpha^{-1}$

$$\tilde{D}^j_{m'm} = \alpha^{-2m} \delta_{m'm}$$

$$g = e^{-i\sigma^3 \phi} = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \Rightarrow g \cdot \tilde{f}_{j,m} = e^{i2m\phi} \tilde{f}_{j,m} \propto |j,m\rangle$$

In QM: angular momentum states $|j,m\rangle$ $J_z|j,m\rangle = \hbar m|j,m\rangle$

$$e^{-iJ_z\phi}|j,m\rangle = e^{-i\hbar m\phi}|j,m\rangle$$

$$(*) = \binom{j+m}{s} \bar{\alpha}^s \bar{\beta}^{j+m-s} (u^s v^{j+m-s}) \binom{j-m}{t} (-\beta)^t \alpha^{j-m-t} u^t v^{j-m-t}$$

$$= \sum_{s,t} \binom{j+m}{s}\binom{j-m}{t} \bar{\alpha}^s \alpha^{j-m-t} \bar{\beta}^{j+m-s} (-\beta)^t u^{s+t} v^{2j-s-t}$$

$(s, t \geq 0)$

$$\Rightarrow \tilde{D}^j_{m'm}(g) = \sum_{s+t=j+m'} \binom{j+m}{s}\binom{j-m}{t} \bar{\alpha}^s \alpha^{j-m-t} \bar{\beta}^{j+m-s} (-\beta)^t$$

$$j = \tfrac{1}{2} \qquad \tilde{D}^{\frac{1}{2}}(g) = \begin{array}{c} \begin{array}{cc} \frac{1}{2} & -\frac{1}{2} \end{array} \\ \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \end{array} = g^\dagger$$

$$m = 1/2 \quad \sum_{\substack{s+t \\ = j+m'}} \binom{1}{s}\binom{0}{0} \bar{\alpha}^s \alpha^0 \bar{\beta}^{1-s} (-\beta)^0$$

$t = 0$

$m' = \frac{1}{2}$ $s = 1$ $\qquad$ $m' = -\frac{1}{2}$ $s = 0$

$$m = -1/2 \quad \sum_{\substack{t = \frac{1}{2} \\ + m'}} \binom{0}{0}\binom{1}{t} \bar{\alpha}^0 \alpha^{1-t} \bar{\beta}^0 (-\beta)^t$$

$m' = \frac{1}{2}$, $t = 1$ $\qquad$ $m' = -\frac{1}{2}$ $t = 0$