Recap:

1. What we did using group algebra & class operators:

$$\hat{C}_i \cdot \hat{C}_j = \sum D_{ij}^{k} \hat{C}_k$$

$$\hat{C}_i \cdot C_j = \sum [D(i)]_{kj} C_k \qquad (\text{c.f. } T(g) \cdot e_i = \sum M_{ji}(g) e_j)$$

$$\boxed{D(i) \text{ is the matrix rep of } \hat{C}_i \text{ on } Z(\mathbb{C}[G])}$$

recall $L^2(G)^{\text{class}}$

① Suppose we find eigen vectors of $\hat{C}_i$ on $Z(\mathbb{C}[G])$

$$\hat{C}_i \phi^\mu = \lambda_i^\mu \phi^\mu \qquad \phi^\mu = \sum \varphi^\mu(C_j) C_j \equiv \varphi_j^\mu C_j$$

what does it mean ?

$$\hat{C}_i \cdot \phi^\mu = \hat{C}_i \cdot \sum \varphi_j^\mu C_j = \sum_j \varphi_j^\mu \hat{C}_i C_j = \sum_{jk} \varphi_j^\mu [D(i)]_{kj} C_k$$

$$\lambda_i^\mu \phi^\mu = \lambda_i^\mu \sum_k \varphi_k^\mu C_k$$

$$\Rightarrow \sum_j [D(i)]_{kj} \varphi_j^\mu = \lambda_i^\mu \varphi_k^\mu \quad \text{, or} \quad \overleftrightarrow{D(i)} \cdot \vec{\varphi}^\mu = \lambda_i^\mu \vec{\varphi}^\mu$$

$\Rightarrow$ $\vec{\varphi}^\mu = (\varphi_1^\mu, \cdots \varphi_r^\mu)^T$ are eigenvectors of $D(i)$

and the corresponding eigenvector of $C_i$:

$$\phi^\mu = \sum_j \varphi_j^\mu C_j$$

what's the point ? what's the meaning of solving this eigen problem ?

② The eigen values:

$$\hat{C}_i \phi^\mu = \lambda_i^\mu \phi^\mu .$$

$\hat{C}_i$ intertwiners, then

Schur's lemma tells us, on an irrep

$$\hat{C}_i^{(\mu)} = \frac{m_i}{n_\mu} \chi^\mu([C_i]) \cdot \mathbb{1}_{V^\mu} =: \alpha_i^\mu \mathbb{1}_{V^\mu}$$

$$(\mathrm{Tr}\ LHS = m_i \cdot \chi([C_i]))$$

If we take trace of $C_i C_j = \Sigma [D(i)]_{kj} C_k$ on the <u>same irrep</u>:

$$LHS = \mathrm{Tr}\ C_i^\mu C_j^\mu = \mathrm{Tr}\ [C_j^\mu \cdot \alpha_i^\mu \mathbb{1}] = \alpha_i^\mu m_j \chi_\mu([C_j])$$

$$RHS = \Sigma [D(i)]_{kj} \mathrm{Tr}\ C_k^\mu = \Sigma [D(i)]_{kj} m_k \chi_\mu([C_k])$$

$$\Rightarrow \quad D(i)^T \vec{\chi}_\mu = \alpha_i^\mu \vec{\chi}_\mu \quad \left([D(i)]_{kj} = D_{ij}^k\right)$$

$$\vec{\chi}_\mu = \begin{pmatrix} m_1 \chi_\mu([C_1]) \\ \vdots \\ m_r \chi_\mu([C_r]) \end{pmatrix} \quad \text{up to normalization}$$

Thus, $\alpha_i^\mu$ are just eigen values $\lambda_i^\mu$, i.e.

$$\lambda_i^\mu = \frac{m_i}{n_\mu} \chi_\mu([C_i])$$

Compare to the above relation: $\vec{\phi}_\mu$ and $\vec{\chi}_\mu$ are just right and left eigenvectors of $D(i)$ with the same eigenvalues $\lambda_i^\mu = \frac{m_i}{n_\mu} \chi_\mu([C_i])$

(if we define $[D(i)]_{kj} = D_{ij}^k$ )

③ The (right) eigen vectors:

we know $\lambda_j^{\mu}$, $\lambda_j^{\nu}$ can not be degenerate for $\forall j$

because otherwise $\chi_\mu$, $\chi_\nu$ not orthogonal.

$\Rightarrow$ we obtain $r$ (right) eigenvectors of $C_i$

$$C_i \phi^{\mu} = \lambda_i^{\mu} \phi^{\mu}.$$

$\phi^{\mu}$ are (proportional to) primitive idempotent on $Z(\mathbb{C}[G])$. (all one-dimensional)

$$\tilde{\phi}^{\mu} \tilde{\phi}^{\nu} = \delta_{\mu\nu} \tilde{\phi}^{\mu} \qquad (\tilde{\phi}^{\mu} = \frac{1}{c}\phi^{\mu}. \text{ if } \phi^{\mu 2} = c\phi^{\mu})$$

What are they? We know

$$V^{-1} D(i)^T V = \Lambda \quad . \quad \Lambda = \mathrm{diag}(\lambda_i^{\mu_1} \cdots \lambda_i^{\mu_r})$$

assume $V = \begin{pmatrix} m_1 \chi^{\mu_1}(C_1) & \cdots & m_1 \chi^{\mu_r}(C_1) \\ \vdots & & \vdots \\ m_r \chi^{\mu_1}(C_r) & & m_r \chi^{\mu_r}(C_r) \end{pmatrix}$

right eigenvectors $V^T D(i) \underline{\underline{(V^{-1})^T}} = \Lambda$

only need to inverse $V$. and take the row vectors

It is actually quite simple using orthogonality of characters:

$$\frac{1}{|G|} \sum_{i=1}^{r} m_i \overline{\chi^{\mu}(C_i)} \chi^{\nu}(C_i) = \delta_{\mu\nu}$$

$$\left( \frac{m_i}{|G|} \sum_{\mu=1}^{r} \overline{\chi^{\mu}(C_i)} \chi^{\mu}(C_j) = \delta_{ij} \right)$$

$$V^{-1} = \frac{1}{|G|}\begin{pmatrix} \overline{\chi^{\mu_1}(C_1)} & \cdots & \overline{\chi^{\mu_1}(C_r)} \\ \overline{\chi^{\mu_2}(C_1)} & \cdots & \overline{\chi^{\mu_2}(C_r)} \\ \vdots & & \\ \overline{\chi^{\mu_r}(C_1)} & \text{—} & \overline{\chi^{\mu_r}(C_r)} \end{pmatrix}$$

So the corresponding right eigenvectors are just

$$\varphi^{\mu_i} = \frac{1}{|G|}\left(\overline{\chi^{\mu_i}(C_1)}, \cdots, \overline{\chi^{\mu_i}(C_r)}\right)^T$$

on $\mathbb{C}[G]$ it corresponds to $\phi^{\mu_i} = \frac{1}{|G|}\sum_i \overline{\chi^{\mu_i}(C_i)} \cdot C_i$

$$= \frac{1}{|G|}\sum_g \overline{\chi^{\mu}(g)}\, g$$

The normalization issue is fixed by $\tilde{\phi}^2 = \tilde{\phi}$.

$$\phi^{\mu_i\,2} = \frac{1}{|G|^2}\sum_{g,h} \overline{\chi(g)}\,\overline{\chi(h)}\, gh \overset{k=gh}{=} \frac{1}{|G|^2}\sum_{k,g} \underbrace{\overline{\chi(g)}\,\overline{\chi(g^{-1}\cdot k)}}_{\substack{(\text{HW 08. P25}) \\ = \frac{|G|}{n_\mu}\overline{\chi^{\mu}(k)}}}\, k$$

$$= \frac{1}{|G|}\frac{1}{n_\mu}\sum_k \overline{\chi^{\mu}(k)}\, k$$

$$= \frac{1}{n_\mu}\cdot \phi^{\mu_i}$$

$\Rightarrow$ $\tilde{\phi}^{\mu_i} = \frac{n_\mu}{|G|}\sum_i \overline{\chi^{\mu_i}(C_i)}\, C_i$ then $\tilde{\phi}^2 = \tilde{\phi}$.

Recall earlier definitions of projectors:

Peter-Weyl / ortho. mat. element:

$$\int_G dg\, \overline{\mathcal{U}^{\mu}_{ij}(g)}\, \mathcal{U}^{\nu}_{kl}(g) = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ik}\delta_{jl}$$

define projectors on $(T, V)$:

$$P^{\mu}_{ij} = n_\mu \int_G dg\, \overline{M^{(\mu)}_{ij}(g)}\, T(g) \in \mathrm{End}(V)$$ , which satisfy

$$P^{\mu}_{ij} P^{\nu}_{kl} = \delta_{\mu\nu} \delta_{jk} P^{\nu}_{il}$$

Transforms under group action as

$$T(h) P^{\mu}_{ij} \psi = \sum_{k=1}^{n_\mu} M^{\mu}_{ki}(h) P^{\mu}_{kj} \psi$$

i.e. $\{ P^{\mu}_{ij} \psi \mid i = 1, \cdots n_\mu \}$ span irrep $V^\mu$

Taking trace $P^\mu = \sum_{i=1}^{n_\mu} P^{\mu}_{ii}$ . then

$$P^\mu = n_\mu \int_G \overline{\chi^\mu(g)}\, T(g)\, dg .$$

satisfying $P^\mu \cdot P^\nu = \delta_{\mu\nu} P^\nu$ and transforms as

$$T(h) P^\mu \psi = \sum_{i,k=1}^{n_\mu} M^{\mu}_{ki}(h) P^{\mu}_{ki} \psi$$

These are projectors onto distinct isotypic components

Now. we think of the representation on $\mathbb{C}[G]$:

$$P^\mu = \frac{n_\mu}{|G|} \sum_g \overline{\chi^\mu(g)} \cdot g = \tilde{\phi}^\mu .$$

2. Now we examine the example of $S_3$ again.

① We define class operators

$$C_1 = E$$

$$C_2 = (12) + (13) + (23)$$

$$C_3 = (123) + (132)$$

② Work out the $D_{ij}^k$ : $[D(i)]_{kj} = D_{ij}^k$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $C_1$ | $C_1$ | $C_2$ | $C_3$ |
| $C_2$ | $C_2$ | $3(C_1+C_3)$ | $2C_2$ |
| $C_3$ | $C_3$ | $2C_2$ | $2C_1+C_3$ |

$$D(1) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad D(2) = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix} \qquad D(3) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

We know $C_2$ is by itself CSCO-I

$$\lambda_2^{\mu_i} = \begin{cases} 3 \\ -3 \\ 0 \end{cases} \qquad \begin{matrix} (1, 1, 1)^T \\ (1, -1, 1)^T \\ (2, 0, -1)^T \end{matrix} \qquad \lambda_i^{\mu} = \frac{m_i \chi_\mu([C_i])}{n_\mu}$$

$$\lambda_2^{\mu} = \frac{3}{n_\mu} \chi_\mu([C_2])$$

$$\phi^\mu \propto (\overline{\chi_\mu[C_1]}, \overline{\chi_\mu[C_2]}, \overline{\chi_\mu[C_3]})^T$$

$n_\mu$ is again fixed by ON of characters.

$$n_{triv} = n_{sgn} = 1, \quad n_{std} = 2$$

After some work. we construct the character table

| $S_3$ | $1[()]$ | $3[(12)]$ | $2[(123)]$ |
|---|---|---|---|
| $V_{triv}$ | 1 | 1 | 1 |
| $V_{sgn}$ | 1 | -1 | 1 |
| $V_{std}$ | 2 | 0 | -1 |

The projectors $P^{\mu} = \frac{n_\mu}{|G|} \sum_i \overline{\chi^{\mu}(C_i)}\, C_i$

$$P^{triv} = \frac{1}{6}(C_1 + C_2 + C_3)$$

$$P^{sgn} = \frac{1}{6}(C_1 - C_2 + C_3)$$

$$P^{std} = \frac{1}{3}(2C_1 - C_3)$$

We haven shown previously

$$g \cdot P^{triv} = P^{triv}$$

$$g \cdot P^{sgn} = sgn(g) \cdot P^{sgn}$$

$$P^{std} = \frac{1}{3}(2E - (123) - (132))$$

$$(12)\, P^{std} = \frac{1}{3}(2(12) - (23) - (13)) \neq \alpha P^{std}.$$

Because $P^{std}$ projects onto 2d. space.

$$\left( T(h) P^{\mu} \psi = \sum_{i,k=1}^{n_\mu} M^{\mu}_{ki}(h) P^{\mu}_{ki} \psi \right).$$

How to find each basis state? We split further

using $S_2 \subset S_3$ $\quad P_+ = \frac{1}{2}(e + (12))$ $\quad P_- = \frac{1}{2}(e - (12))$

$$\begin{cases} P^{std}_+ = P^{std} P_+ \\ P^{std}_- = P^{std} P_- \end{cases} \quad \text{then } P^{std} = P^{std}_+ + P^{std}_- \text{ . and } P^{std}_+ P^{std}_- = 0$$

## §14 Representation of $S_n$

(Miller, book Chap 4)

see also 陈金全

contains all proofs of the statements below.

References:

① Moore § 11.15

☆ ② More details in

Miller, "Symmetry groups and their applications"

Chap. 4. Symmetric group rep.

③ 陈金全. not very easy to follow.

---

Basics of $S_n$:

$(i_1, i_2, \cdots i_r) \sim (j_1, j_2, \cdots j_r)$

$r$-cycles are conjugate

$S_n$ irreps are defined by vectors

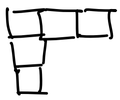
$\vec{\ell} = (\ell_1, \ell_2 \cdots \ell_n)$

$\ell_i$ the number of $i$-cycles

conj. classes $\Leftrightarrow$ Young diagrams.

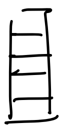
partition (123)(4)(5)

(3, 1. 1)



(12)(45)

( )



We construct the irreps of $S_n$ using group algebra. which means finding projectors (idempotents) onto different isotypic components. (or better irrep basis)

For 1D irreps, we can generalize from last lecture that $c = \frac{n_\mu}{|G|} \Sigma \chi_\mu(g) g$

① trivial irrep:

$$c = \frac{1}{n!} \sum_{s \in S_n} s, \quad c^2 = c$$

$$cs = sc = c \quad (\forall s \in S_n)$$

The subspace $\{\lambda c\}$ is the trivial irrep

$$L(s) \cdot c = sc = c$$

② the sgn irrep:

$$c = \frac{1}{n!} \sum_{s \in S_n} \mathrm{sgn}(s) \cdot s$$

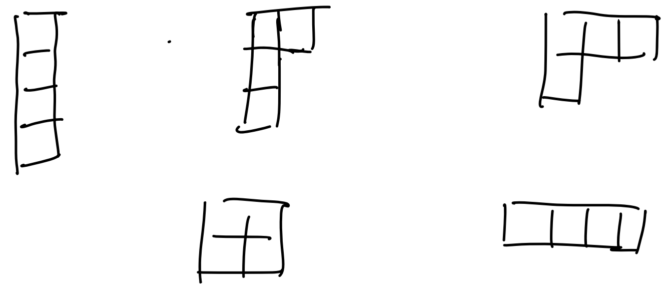
$$cs = sc = \mathrm{sgn}(s) \cdot c \quad (\forall s \in S_n)$$

$$L(s) \cdot c = \mathrm{sgn}(s) \cdot c.$$

③ How to find projectors / idempotents onto other irreps?

$\Rightarrow$ use Young diagrams & Young tableaux.

$S_4$.



Note that we now know # Conj. cls = # irreps. these diagrams also labels irreps. as we shall see later with:

Young tableaux = Young diagram + numbers

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

$n!$ tableaux for a diagram

$$\boxed{\text{Standard tableaus: numbers increase within rows and columns}}$$

Given a tableau T. we define two sets of permutations $R(T)$, $C(T)$

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

$R(T) = \{e, (12), (13), (23), (123), (132)\}$

$C(T) = \{e, (14)\}$

$R(T) \cap C(T) = \{e\}$,

and for $\forall p \in R(T), q \in C(T)$. $pq$ is unique

if $p' \in R(T), q' \in C(T)$, and $pq = p'q'$. then

$$p' q' = p q \Leftrightarrow \underset{C(T)}{\underset{\uparrow}{q (q')^{-1}}} = \underset{R(T)}{\underset{\uparrow}{p^{-1} \cdot p'}} = e \quad \Rightarrow p = p', \; q = q'$$

Then we construct two elements of $R_{S_n} =: R_n$

$$P = \sum_{p \in R(T)} p \qquad Q = \sum_{q \in C(T)} \epsilon(q) \cdot q \qquad \left( \epsilon(q) = \mathrm{sgn}(q) \right)$$

Define $c = PQ = \sum_{\substack{p \in R(T) \\ q \in C(T)}} \epsilon(q)\, p q \neq 0$ ($\because$ all $pq$'s are unique)

(also called Young symmetrizers)

Theorem 1. $c = PQ$ corresponding to a tableau $T$

is essentially idempotent (i.e. $c^2 = \lambda c$)

The invariant subspace $R_n c := \{ g c, \; \forall g \in R_n \}$

yields an irrep of $S_n$.

$\Rightarrow$ $\tilde{c} = \lambda^{-1} c$. then $\tilde{c}^2 = \tilde{c}$ idempotent.

That also means for $T' \neq T$. $c \cdot c' = 0$

( recall projectors $P^\mu P^\nu = \delta_{\mu\nu} P^\mu$ )

Theorem 2. The dimension $f$ of the irrep

corresponds to a diagram is the number of

standard tableaux $\{ T_i, \; i = 1, \cdots f \}$

Example $S_3$

① $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ $\quad f = 1 \quad c_{(3)} = \sum_{s \in S_3} s$

② $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\quad f = 1 \quad c_{(1,1,1)} = \sum_{s \in S_3} \epsilon(s)\, s$

③ $T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad f = 2 \quad R(T_1) = \{e, (12)\}, \; Q(T_1) = \{e, (13)\}$

$$C(T_1) = (e + (12))(e - (13))$$
$$= e + (12) - (13) - (132)$$

$T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad R(T_2) = \{e, (13)\}, \; Q(T_2) = \{e, (12)\}$

$$C(T_2) = e + (13) - (12) - (123)$$

same as what we had before.

The normalization factor:

$$C(T)^2 = \lambda(T)\, C(T) \quad , \text{ then}$$

$$\lambda(T) = \frac{n!}{f} \quad \left(\text{i.e. } \frac{n_\mu}{|G|} \text{ from before}\right).$$

$f$ is found using the "hook length formula"

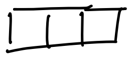
$$f = \frac{n!}{\prod_b h(b)}$$

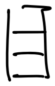
where $h(b)$ is the hook length.

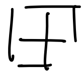

$h(b_1) = 8$

$h(b_2) = 4$

So back to $S_3$:

 $f = \frac{3!}{1\times 2\times 3} = 1$



 $f = \frac{3!}{1\times 3\times 1} = 2$

$$\begin{cases} \tilde{C}_{triv} = \frac{1}{6}\sum s \\ \tilde{C}_{sgn} = \frac{1}{6}\sum \epsilon(s)\, s \\ \tilde{C}_{std} = \begin{cases} \frac{1}{3}[e + (12) - (13) - (132)] =: \tilde{C}_1 \\ \frac{1}{3}[e - (12) + (13) - (123)] =: \tilde{C}_2 \end{cases} \end{cases}$$

We try to come up with a matrix rep of $V_{std}$.

in on $\mathbb{C}[S_3]$:

$R_{S_3}\cdot \tilde{C}_1$:

$$e\cdot \tilde{C}_1 = \tilde{C}_1 =: v_1$$

$$(12)\cdot \tilde{C}_1 = \frac{1}{3}\left((12) - (132) + e - (13)\right) = \tilde{C}_1$$

$$(13)\cdot \tilde{C}_1 = \frac{1}{3}\left((13) - e + (123) - (23)\right) =: v_2$$

$$(23)\cdot \tilde{C}_1 = -v_1 - v_2$$

$$(123)\cdot \tilde{C}_1 = v_2$$

$$(132)\cdot \tilde{C}_1 = -v_1 - v_2$$

Matrix rep. of $V = \mathrm{span}\{v_1, v_2\}$

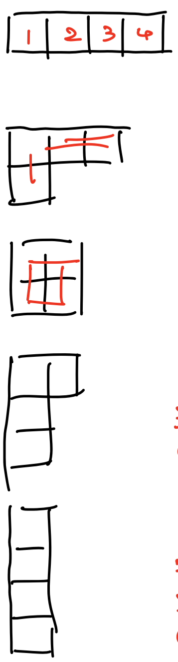
$$\begin{cases} (12)\cdot v_1 = v_1 \\ (12)\cdot v_2 = (12)\left((13)\cdot v_1\right) \end{cases}$$

$$= (132)\cdot v_1 = -v_1 - v_2$$

$$M[(12)] = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

## Example : Character table of $S_4$.

1. Conjugacy classes ?  2. irreps ? = # conj. class.



$(4)$  $f = \frac{4!}{\prod_b h(b)} = 1$  1

$(3)(1)$  $f = \frac{4!}{4\times 2} = 3$  3

| 1 | 2 | 3 |
| 4 |

| 1 | 2 | 4 |
| 3 |

| 1 | 3 | 4 |
| 2 |

$(2)^2$  $f = \frac{4!}{3\times 2\times 2} = 2$  2

| 1 | 2 |
| 3 | 4 |

| 1 | 3 |
| 2 | 4 |

$(2)(1)^2$  3

1 2 / 3 / 4   1 3 / 2 / 4   1 4 / 2 / 3

$(1)^4$  1

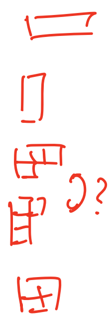
1 / 2 / 3 / 4

$$|G| = \sum_\mu n_\mu^2$$

$$1+3^2+2^2+3^2+1 = 24 = 4!$$

$\binom{4}{2}=6$  $\binom{4}{2}/2$  $\binom{4}{3}\cdot 2$

| | E | 6 [(12)] | 3 [(12)(34)] | 8 [(123)] | 6 [(1234)] |
|---|---|---|---|---|---|
| $V^+$ | 1 | 1 | 1 | 1 | 1 |
| $V^-$ | 1 | −1 | 1 | 1 | −1 |
| $V^\perp$ | 3 | 1 | −1 | 0 | −1 |
| $V^-\otimes V^\perp$ | 3 | −1 | −1 | 0 | 1 |
| $V^2$ | 2 | 0 | 2 | −1 | 0 |



??