What we know so far:

① ortho. matrix elements (Peter-Weyl)

$$\int_G dg\, \overline{M^{(\mu)}_{ij}(g)}\, M^{(\nu)}_{kl}(g) \left(\equiv \langle M^{(\mu)}_{ij} . M^{(\nu)}_{kl} \rangle\right) = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ik}\delta_{jl}$$

② ~~taking~~ trace of ①:

$$\int_G dg\, \overline{\chi^{(\mu)}(g)}\, \chi^{(\nu)}(g) = \delta_{\mu\nu}$$

③ $\{\chi^{(\mu)}\}$ ON basis of $L^2(G)^{\text{class}}$. spanned by $\{\delta_{C_i}\}$

$\forall f \in L^2(G)^{\text{class}}$

$$\left(f(g) = \sum_{j=1}^{r} \alpha_j \delta_{C_j}(g) \right)$$

with $\alpha_j = f(C_j)$

$\Rightarrow$ number of irreps = number of conj. classes $= r$

④ character table of finite groups:

$$\frac{1}{|G|}\sum_g m_i\, \overline{\chi^\mu(C_i)}\, \chi^\nu(C_i) = \delta_{\mu\nu}$$

$$S_{\mu i} = \sqrt{\frac{m_i}{|G|}}\, \chi^\mu(C_i)$$

then $\overline{S_{\mu i}}\, S_{\nu i} = \delta_{\mu\nu}$, or $S_{\nu i}(S^+)_{i\mu} = \delta_{\mu\nu}$. S unitary

columns also orthogonal. i.e.

$$(S^T)_{i\mu}(\overline{S})_{\mu j} = \delta_{ij}, \quad S_{\mu i}\, \overline{S}_{\mu j} = \delta_{ij}$$

$$\sum_\mu^r \overline{\chi^\mu(C_i)}\, \chi^\mu(C_j) = \frac{|G|}{m_i}\delta_{ij}$$

③ given a character table. for any rep V.

we can work out the isotypic decomposition

$$V \cong \oplus a_\mu V^{(\mu)}$$

$$\chi_V = \sum a_\mu \chi^{(\mu)}$$

$$\Rightarrow \langle \chi_\mu, \chi_V \rangle = a_\mu$$

8.13 Decomposition of tensor products of representations.

V carries space of dim n, basis $\{v_1, \cdots v_n\}$

W m $\{w_1, \cdots w_m\}$

$V \otimes W$. dim $n \cdot m$ basis $\{v_i \otimes w_j \quad 1 \leq i \leq n,\ 1 \leq j \leq m\}$

$$\sum_i a_i v_i \otimes \sum_j b_j w_j = \sum_{ij} a_i b_j \, v_i \otimes w_j$$

G-action $\quad g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$

rep. $\quad (T_1 \otimes T_2)(g)(v \otimes w) := T_1(g) \cdot v \otimes T_2(g) w.$

mat. rep. $\quad (M_1 \otimes M_2)(g)_{ia,jb} = [M_1(g)]_{ij} [M_2(g)]_{ab}$

character $\quad \chi_{T_1 \otimes T_2} = \chi_{T_1} \cdot \chi_{T_2}$

① particle of spin j_1

j_2 $\Rightarrow V^{j_1} \otimes V^{j_2}$

$$\cong \oplus a_{j_3} V^{j_3}$$

② many-particle system, local Hilbert space

$\mathcal{H}_i$ spin 1/2 fermion $= \{\phi, \uparrow, \downarrow, \uparrow\downarrow\}$

$$\mathcal{H} = \otimes_i \mathcal{H}_i \Rightarrow \oplus_i a_i \mathcal{H}^{\text{conf}_i}$$

N.S.

$\underline{G} \otimes U(1) \otimes SU(2)$

space group

Let (V_1, T_1) and (V_2, T_2) be two representations with isotypic decompositions (over field k)

$$V_1 = \oplus a_\mu V^\mu \qquad V_2 = \oplus b_\nu V^\nu$$

$$V_1 \otimes V_2 = \oplus \underset{\sim}{a_\mu b_\nu} V^\mu \otimes V^\nu$$

$$V^\mu \otimes V^\nu \cong \oplus_\lambda \underline{\underline{N_{\mu\nu}^\lambda}} V^\lambda \qquad \left(\oplus D_{\mu\nu}^\lambda \otimes V^\lambda \right)$$

$$\underline{\underline{\chi_\mu \cdot \chi_\nu}} = \sum_\lambda N_{\mu\nu}^\lambda \chi_\lambda$$

fusion coefficient

Clebsch-Gordan for SU(2)

$$N_{\mu\nu}^\lambda = \langle \chi_\lambda, \chi_\mu \cdot \chi_\nu \rangle$$

for Finite groups

$$N_{\mu\nu}^\lambda = \frac{1}{|G|} \sum_{g \in G} \chi_\mu(g) \chi_\nu(g) \overline{\chi_\lambda(g)}$$

$m_i = |C_i|$

$$= \frac{1}{|G|} \sum_{\{C_i\}} m_i \chi_\mu(C_i) \chi_\nu(C_i) \overline{\chi_\lambda(C_i)}$$

$$N_{\mu\nu}^\lambda = N_{\nu\mu}^\lambda \qquad \left(V^\mu \otimes V^\nu \cong V^\nu \otimes V^\mu \right)$$

<u>Examples</u>. 1. ρ_m of $\mathbb{Z}_N$ $\qquad \rho_m(j) = \left(e^{i\frac{2\pi}{N}j}\right)^m$

$$\rho_m \otimes \rho_n \cong \rho_{m+n \bmod N}$$

$$N_{mn}^\lambda = \frac{1}{N} \sum_{l=0}^{N-1} \underline{e^{i\frac{2\pi}{N}\overline{(m+n)}l} \, e^{-i\frac{2\pi}{N}\cdot \lambda l}}$$

$$= \delta_{\overline{m+n}, \lambda}$$

2. irreps of S_3.

$$V^{1+} \otimes V^{\mu} \cong \oplus_{\lambda} N_{1^+,\mu}^{\lambda} V^{\lambda}$$

$$N_{1^+,\mu}^{\lambda} = \frac{1}{|G|} \sum m_i \chi_{\mu}(C_i) \overline{\chi_{\lambda}(C_i)}$$

$$= \delta_{\mu\lambda}$$

$$\bigoplus_{\lambda} \delta_{\mu\lambda} V^{\lambda} = V^{\mu}$$

$$\Rightarrow V^{1+} \otimes V^{\mu} \cong V^{\mu}$$

<u>check</u>

$$V^{-} \otimes V^{-} \cong V^{+}$$

$$V^{-} \otimes V^{2} \cong V^{2}$$

$$V^{2} \otimes V^{2} \cong V^{+} \oplus V^{-} \oplus V^{2}$$

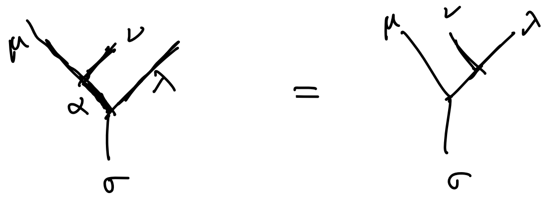
$$(V^{\mu} \otimes V^{\nu}) \otimes V^{\lambda} \cong V^{\mu} \otimes (V^{\nu} \otimes V^{\lambda})$$

$$\text{LHS} \cong \oplus_{\alpha} D_{\mu\nu}^{\alpha} V^{\alpha} \otimes V^{\lambda}$$

$$\cong \oplus_{\sigma} (\oplus_{\alpha} D_{\mu\nu}^{\alpha} \otimes D_{\alpha\lambda}^{\sigma}) \otimes V^{\sigma} \cong \oplus_{\sigma} (\oplus_{\beta} D_{\nu\lambda}^{\beta} \otimes D_{\mu\beta}^{\sigma}) \otimes V^{\sigma}$$

$$\sum_{\alpha} N_{\mu\nu}^{\alpha} N_{\alpha\lambda}^{\sigma} = \sum_{\beta} N_{\mu\beta}^{\sigma} N_{\nu\sigma}^{\beta}$$

"F-move"



digression : "Category theory"

TQFT / anyons / topo. quantum computation

$(x \otimes y) \otimes (z \otimes w) \rightarrow$ pentagon relation

(ref. PRB 100, 115147)

Summary of key results

① unitary rep. of compact G.

$$\langle u^{\mu_1}_{i_1 j_1}, u^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$$

complete, orthogonal basis of $L^2(G)$.

② (Peter-Weyl) $L^2(G) \cong \bigoplus_\mu \mathrm{End}(V^\mu)$

$$\iota : \bigoplus_\mu \mathrm{End}(V^\mu) \longrightarrow L^2(G)$$

$$\bigoplus_\mu S_\mu \longmapsto \sum_\mu \varphi_{S_\mu}$$

$$\varphi_{S_\mu} := \mathrm{Tr}_{V_\mu}(S\,T(g^{-1}))$$

↪ finite G: $\boxed{|G| = \sum_\mu n_\mu^2}$ $(n_\mu = \dim V^\mu)$

③ characters.

$$\int_G \overline{\chi^{(\mu)}(g)}\, \chi^{(\nu)}(g)\, dg = \delta_{\mu\nu}$$

ON basis of $L^2(G)^{\text{class}}$.

④ $V \cong \bigoplus_\mu a_\mu V^{(\mu)}$

$$a_\mu = \int_G \overline{\chi^{(\mu)}}(g)\, \chi_V(g)\, dg = \langle \chi^{(\mu)}, \chi_V \rangle$$

reg. rep. $a_\mu = \langle \chi^\mu, \chi \rangle = \frac{1}{|G|}(\dim n_\mu) \cdot |G|$

$= \dim n_\mu$

⑤ # irreps = # conj. class.

		C_i	- - - -
irreps	Γ_1		
	Γ_2		
	$\vdots$		

rows: $\frac{1}{|G|}\sum_{\{C_i\}} |C_i|\, \chi_\mu(C_i)\, \overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$

columns: $\sum_\mu \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i}\delta_{ij}$

8.13 Group algebra of finite groups.

Refs.

① Fulton & Harris. Representation theory. (GTM 129)

Sec. 3.4.

☆ ② Miller. "Symmetry groups and their applications".

Chap 3.

Chap 4 symmetric group rep.

☆ ③ 陈金全. 第二章 群表示基础

"群元既是算符. 又是基矢".

representation : $G \longrightarrow GL(V)$

Introduce a new vector space R_G (group ring)

Ring : set with $+$ and $\times$

① $+$: commutative ; 0 identity ; $-a$ inverse

② $\times$: (monoid). associative, identity

③ $a \times (b+c) = a \times b + a \times c$. distributive

§. 13.1. group algebra

Let G be a finite group of order n.

Define n-dim vector space R_G with basis ($R[G]$)

$\{ g . g \in G \}$

$$x = \sum_{g \in G} x(g) \cdot g \qquad x \in R_G \qquad x(g) \in \mathbb{C}.$$

$x = y$ iff $\forall g \in G$. $x(g) = y(g)$

$$x + y = \sum_{g \in G} x(g) \cdot g + \sum_{g \in G} y(g) \cdot g = \sum (x(g) + y(g)) g$$

$$\alpha x = \sum_{g \in G} \alpha x(g) \cdot g \qquad \leftarrow \text{promote ring to algebra} \qquad \alpha \in \mathbb{C}.$$

$$0 = \sum_{g \in G} 0 \cdot g$$

$$x y = \Big(\sum_{g \in G} x(g) \cdot g \Big) \Big(\sum_{h \in G} y(h) h \Big) = \sum_{g, h} x(g) y(h) g h$$

$$\overset{gh \to k}{=} \sum_k \Big(\sum_g x(g) y(g^{-1} k) \Big) \cdot k \equiv \sum_k xy(k) \cdot k$$

$$xy(k) = \sum_g x(g) y(g^{-1} k) \qquad \text{convolution product}$$

$$\Big((f * g)(t) = \int f(\tau) g(t - \tau) d\tau \Big)$$

$\Rightarrow R_G$ is a group ring / group algebra $K[G]$

$x(g) \in R$ $\qquad$ $x(g) \in \mathbb{C}$ $\qquad$ $\mathbb{C}[G]$

Review of basic ideas of rep. theory:

Regular representation: $G \times G$

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1})$$

$$(g_1, g_2)x = g_1 x g_2^{-1} \qquad \left(\begin{array}{l} g_i \in G \\ x \in R_G \end{array} \right)$$

now consider L & R : $G \to GL(R_G)$

$\hookrightarrow$ restrict to subgroups $G \times \{1\}$. or $\{1\} \times G$.

LRR: $L(g) \cdot x = gx$

RRR: $R(g) x = x g^{-1}$

$$L(h) \cdot x = L(h) \cdot \sum_g x(g) \cdot g = \sum_g x(g)(hg) = \sum_g x(h^{-1}g) \cdot g$$

$$\equiv \sum_g [L(h) \cdot x](g) \cdot g$$

(View x also as functions on G. $x: G \to \mathbb{C}$, $g \mapsto x(g)$)

$\Rightarrow$ $[L(h) \cdot x](g) = x(h^{-1}g)$

$([R(h) \cdot x](g) = x(g \cdot h))$

Define inner product

$$\langle x, y \rangle = \int_G \overline{x(g)}\, y(g)\, dg \overset{\text{finite}}{=} \frac{1}{|G|} \sum_g \overline{x(g)}\, y(g)$$

$\Rightarrow \langle L(h)x, L(h)y \rangle = \langle x, y \rangle$ unitary reps

We will use $L(h)$, h, δ_h etc. interchangeably

$$\underline{h} = \sum_g h(g)\cdot g = 1\cdot h \quad \Rightarrow \quad \underset{\delta_h}{\overset{\parallel}{h(g)}} = \begin{cases} 1 & g = h \\ 0 & \text{otherwise} \end{cases}$$

(recover $\underline{\delta_h}$ from before)

$$\underline{\delta_h \cdot \delta_g} = \sum_k \Big(\sum_l \delta_h(l)\, \delta_g(l^{-1}\cdot k)\Big)\cdot k = 1\cdot(hg) = \underline{\delta_{hg}}$$

$l = h$.

$l^{-1} k = g \quad k = hg$

see h as left action: $\underline{L(h)\delta_g(g')} = \delta_g(h^{-1}\cdot g') = \delta_{hg}(g')$

$$\underline{L(h)\delta_g = \delta_{hg}}$$

group elements can be viewed both as operators and vectors on R_G

Also. expand the class function on R_G:

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{\delta_{C_i}} = \sum_{g\in G} \underline{\delta_{C_i}(g)}\cdot g = \underline{\underline{\sum_{g\in C_i} g}}$$

(or view as class operators C_i)

$\forall h \in G$: $h C_i h^{-1} = \sum_{g\in C_i} h g h^{-1} = \sum_{g'\in C_i} g' = C_i$. $\underline{\underline{C_i \text{ commutes with } \forall h \in G}}$

$\{C_i\}$ spans the center of $\mathbb{C}[G]$ $Z(\mathbb{C}[G])$

8.13.2. Projectors onto invariant subspaces

$$V = \bigoplus_i W^i \quad \text{invariant subspace.}$$

Suppose. $V = W \oplus W^{\perp}$

Define projector P onto W.

$\forall x \in V$. $\quad x = w + w^{\perp} \quad w \in \underline{W}$, $w^{\perp} \in W^{\perp}$

then $\underline{Px = w} \qquad gw \in W$

$$\forall g \in G: \quad g(Px) = gw = P(\underline{gw}) \overset{P(gw^{\perp})=0}{=} Pg(w + \underline{w^{\perp}}) = Pg\,x$$

$$\Rightarrow \underline{gP} = Pg \qquad \underline{\underline{P \text{ also commutes with } \forall g \in G.}}$$

$$\forall x \in R_G: \quad Px = \sum_g x(g) \cdot Pg = \underbrace{\sum x(g)\, g}_{x} Pe = x\, Pe$$

Define $e' = Pe$: $e'^2 = PePe = P^2 e = Pe = e'$ idempotent 幂等元

then the invariant subspace is defined as

$$W = \{ x e' : x \in R_G \} =: R_G \cdot e'$$

$$\{ Px : \forall x \in R_G \}$$

If $P_1 + P_2 = \mathbb{1} \Rightarrow e = \mathbb{1} e = (P_1 + P_2) e = e_1 + e_2$

$P_1 P_2 = 0 \Rightarrow e_1 e_2 = 0$

irreps: e' is primitive, can not be decomposed into $e_1' + e_2'$ $(e_1' \neq 0,\ e_2' \neq 0)$

Both C_i and P commutes with $\forall g \in G$. is it possible to find

P's onto irreps using C_i?

8.13.3 Construction of character table

We've seen a few character tables for simple groups.

But how do we construct the character tables?

We present an algorithm to obtain them

If we can find all the projectors onto irreps, or equivalently all the idempotents.

Some ideas:

Recall previously, a Hamiltonian H is an intertwiner. $[H, T(G)] = 0$

The eigenvectors $\{\psi_\mu\}$ span an invariant subspace W of the representation space $L^2(G)$:

$$H\psi_\mu = E_\mu \psi_\mu$$

$$H \underline{T(g)\psi_\mu} = T(g) H \psi_\mu = E_\mu \underline{T(g)\psi_\mu} \quad \forall g \in G$$

$T(g)\psi_\mu \in W. (\forall g \in G) \Rightarrow W$ is an invariant subspace, i.e. a representation space

$$V \cong \oplus W^\mu$$

If W^μ is still reducible, find another operator that satisfies $[O, T(g)] = 0$ $(\forall g \in G)$

With a complete set of commuting operators (CSCO), we can achieve a complete reduction of representations / find all irreps!

This is an idea explored systematically by 陈金全 (南大).

① 陈金全.《群表示论的新途径》

② English translation: Group representation theory for physicists. 2nd. Ed. World Scientific, 2002

③ The representation group and its application to space groups

RMP 57, 211 (1985)

First RMP of PRC.

To illustrate the idea, consider a finite group G.

with r conjugacy classes $[C_i]$ $(i=1, \cdots r)$

$|[C_i]| = m_i$. Correspondingly, r irreps V^μ and characters χ_μ

What operator commutes with all elements of R_G

The center of the group algebra $Z[R_G]$

is spanned by the class operators / functions

$\forall x \in Z(R_G)$: $\quad c_i = \sum'_{g \in C_i} g$

$x g x^{-1} = g$

They have the following properties:

① $\forall h \in G.\ [c_i, h] = 0$: $h c_i h^{-1} = \sum_{g \in C_i} h g h^{-1} = c_i$

② $\forall i,j\ \ [c_i, c_j] = 0$: because of ①

③ closed/complete: $c_i c_j = \sum_{k=1}^{r} c_{ij}^k c_k$, $(c_{ij}^k = c_{ji}^k \in \mathbb{N})$ where

c_{ij}^k the class multiplication coefficient, something we can easily compute given a group.

Proof: $\forall h_{i1}, h_{i2} \in C_i$. $\exists g' \in G$, s.t. $h_{i1} = g' h_{i2} g'^{-1}$

$$\sum_{g \in G} g h_{i1} g^{-1} = \sum_g g(g' h_{i2} g'^{-1}) g^{-1} = \sum_g g h_{i2} g^{-1}$$

$m_i = |C_i|$

$g_{ia} \in C_i$

$$\Rightarrow \sum_{g \in G} g c_i g^{-1} = m_i \sum_{g \in G} g g_{ia} g^{-1},$$

$\because$ ①, LHS $= |G| \cdot c_i$

$$\Rightarrow \sum_{g \in G} g g_{ia} g^{-1} = \frac{|G|}{m_i} c_i \quad \Delta$$

S-O theorem / class eq. $|C(g)| = \frac{|G|}{|Z_G(g)|}$

one element on LHS. then full class on RHS

$$① \Rightarrow c_i c_j = \frac{1}{|G|} \sum_{g \in G} g (c_i c_j) g^{-1}$$

Any $g \in c_i c_j$, belongs to some C_k, then RHS contains full C_k

$$\Rightarrow \boxed{c_i c_j = \sum_{k=1}^{r} c_{ij}^k c_k} \qquad (*)$$

(Should they be enough for finding all irreps of a group? Some arguments: we've mentioned before that $\{\delta_{C_i}\}$ is a complete basis for $L^2(G)^{class}$, so is $\{\chi_\mu\}$.)

If we can diagonalize some/all C_i's, and decompose them into projectors / find idempotents.

From an algebraic point of view, Eq. (*) provided us with a set of eigen problems.

$$\hat{C}_i \, \delta_{C_j} = \sum_{k=1}^{r} [C^i]_{jk} \, \delta_{C_k}$$

With $\{\delta_{C_i}\}$ an orthogonal basis: of class algebra

(recall inner product $\langle \delta_{C_j}, \delta_{C_k} \rangle = \frac{1}{|G|} \sum_g \delta_{C_j}(g) \delta_{C_k}(g) = \frac{m_j}{|G|} \delta_{jk}$)

Suppose for $\hat{C}_i$ we find its eigenvectors $\{\phi^\mu\}$

$$\hat{C}_i \, \phi^\mu = \lambda_i^\mu \phi^\mu$$

$\lambda^\mu = \lambda^\nu$, or $\phi^\mu \phi^\nu = 0$

then $\hat{C}_i (\phi^\mu \phi^\nu) = \lambda_i^\mu (\phi^\mu \phi^\nu) \overset{[\hat{C}_i, g] = 0}{=} \lambda^\nu (\phi^\mu \phi^\nu)$, i.e. $\phi^\mu \phi^\nu$ is also an eigen vector associated to λ_i^μ. Assuming λ_i^μ is nondegenerate.

then $\phi^\mu \phi^\nu = \alpha_\mu \delta_{\mu\nu} \phi^\mu$, α_μ some constant $\in \mathbb{C}$. (if $\hat{C}_i$ is a CSCO)

Define $P^\mu = \alpha_\mu^{-1} \phi^\mu$, $P^\mu P^\nu = \delta_{\mu\nu} P^\mu$. $\{P^\mu e\}$ are the primitive idempotents of R_G. $\hookrightarrow$ projectors onto 1D space

and $C_i = \sum_{\mu=1}^{r} \lambda_i^\mu P^\mu$ is actually a linear combination of projectors onto irreps.