Recap:

1. Schur's lemma:

① (T_1, V_1), (T_2, V_2) irreps.

$A: V_1 \to V_2$ s.t.

$$A\, T_1(g) = T_2(g)\, A \quad \text{then}$$

A zero or isomorphism

Prove by realizing $\ker A$, $\operatorname{im} A$ are invariant subspaces.

② $V_1 = V_2 = V$, $A = \lambda \mathbb{1}_V \quad \lambda \in \mathbb{C}$.

Proof by noticing $Av = \lambda v$ must hold for some $\lambda \in \mathbb{C}$. then the eigen vectors span an invariant subspace.

Quantum numbers $=$ irreps

block structure of H ; selection rules etc.

$$\mathcal{H} \cong \bigoplus D_\mu \otimes \mathcal{H}_\mu, \quad T(g) = \mathbb{1} \otimes T(g)^{(\mu)} \Rightarrow H = H^{(\mu)} \otimes \mathbb{1}_{V^{(\mu)}}$$

s.t. $[H, T] = 0$

2. Peter-Weyl: matrix elements of <u>unitary</u> <u>irreps</u> are orthogonal basis of $L^2(G)$

① orthogonality.

Proof by construct intertwiner

$$\tilde{A} = \int_G T^{(\nu)}(g) A T^{(\mu)}(g^{-1}) dg$$

use the fact $\tilde{A} = \lambda \mathbb{1}_{V^{(\mu)}} \delta_{\mu\nu}$

$$\Rightarrow \int_G \overline{M^{(\mu)}_{ij}(g)} M^{(\nu)}_{kl}(g) dg = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik} \delta_{jl}$$

(finite groups $\int_G dg \to \frac{1}{|G|} \sum_g$)

Proofs see Dresselhaus. or any standard textbook.

$$\int_G \overline{\phi^{(\mu)}_{ij}(g)} \phi^{(\nu)}_{kl}(g) dg = \delta_{\mu\nu} \delta_{ik} \delta_{jl}.$$

② completeness.

$W = \text{span}\{\phi\}$ invariant

$$R(h) M^{(\mu)}_{ij}(g) = M^{(\mu)}_{ij}(gh) = \sum M^{(\mu)}_{kj} M^{(\mu)}_{ik}(g)$$

$$R(h) \phi^{(\mu)}_{ij}(g) = \phi^{\mu}_{ij}(gh) = \sum M^{(\mu)}_{kj} \underline{\phi^{(\mu)}_{ik}(g)}$$

$\Rightarrow W^\perp \neq 0$? invariant $\supset V^{(\mu)}$

but $V^{(\mu)}$ is spanned by $\{M\}$. contradiction.

$\Rightarrow W^\perp = 0$

$\Rightarrow W$ complete

$$L^2(G) \cong \hat{\bigoplus} V^{(\mu)} \Rightarrow \text{finite} : |G| = \sum_\mu n_\mu^2$$

8.11 Explicit decomposition of a representation

Let (T, V) be any rep. of a compact group G. Define

$$\underline{P_{ij}^{(\mu)}} := n_\mu \int_G \overline{M_{ij}^{(\mu)}(g)}\, T(g)\, dg$$

M_{ij}^{μ} w.r.t unitary irreps with ON basis of $V^{(\mu)}$.

$$\boxed{P_{ij}^{(\mu)} P_{kl}^{(\nu)} = \delta^{\mu\nu} \delta_{jk} P_{il}^{(\nu)}}$$

$$T(h) P_{ij}^{\mu} = n_\mu T(h) \int_G dg\, \overline{M_{ij}^{(\mu)}(g)}\, T(g)$$

$$= n_\mu \int_G dg\, \overline{M_{ij}^{(\mu)}(g)}\, T(hg)$$

$$\overset{hg \to g}{=} n_\mu \int_G dg\, \underbrace{\overline{M_{ij}^{(\mu)}(h^{-1}g)}}_{M_{ki}^{(\mu)}(h)\, \overline{M_{kj}^{(\mu)}(g)}}\, T(g)$$

$$= \sum_k^{n_\mu} M_{ki}^{\mu}(h)\, P_{kj}^{(\mu)}$$

$$T(h) P_i^{\mu j} = \sum_k M_{ki}^{\mu}(h)\, P_k^{\mu j}$$

$\forall \varphi \in V$. $(P_{ij}^{\mu} \varphi \neq 0)$. then

span $\{ \underline{P_{ij}^{\mu} \varphi}, \ i = 1, \cdots n_\mu \}$ (fix μ, j)

transforms as (T^{μ}, V^{μ})

8.12 Orthogonality relations of characters ; Charater table.

8.12.1 Orthogonality relations —

Recall. a class function on G:

$$f : G \to \mathbb{C}.$$

$f(g) = f(hgh^{-1})$ $\forall g, h \in G$. They span a subspace

$$L^2(G)^{class} \subset L^2(G).$$

<u>Theorem</u> The characters $\{\chi_\mu\}$ is an orthonormal (ON) basis for the vector space of class functions $L^2(G)^{class}$.

<u>Proof</u>. $\int_G dg\, \overline{M^{(\mu)}_{ij}(g)}\, M^{(\nu)}_{kl}(g) = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik} \delta_{jl}$

set $i=j$. $k=l$ & sum over i,k

$$\Rightarrow \int_G dg\, \overline{M^{(\mu)}_{ii}(g)}\, M^{(\nu)}_{kk}(g) = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik}$$

$$\overset{\sum_{i,k}}{\Longrightarrow} \int_G dg\, \overline{\chi^\mu(g)}\, \chi^\nu(g) = \delta_{\mu\nu}$$

$\Rightarrow \{\chi_\mu\}$ ON set

Completeness ?

$$\forall f \in L^2(G) \xrightarrow[\{M^\mu_{ij}\} \text{ complete}]{\text{Peter-Weyl}} f(g) = \sum_{\mu,i,j} \hat{f}^\mu_{ij} M^\mu_{ij}(g)$$

of $f \in L^2(G)^{\text{class}}$ $\quad f(g) = f(hgh^{-1})$

$$\int_G dh\, f(g) = \int_G dh\, f(hgh^{-1})$$

LHS: $= f(g)$

RHS: $\int_G f(hgh^{-1})dh = \sum_{\mu,i,j} \hat{f}^\mu_{ij} \int_G \underbrace{M^\mu_{ij}(hgh^{-1})}_{M^\mu_{ik}(h) M^\mu_{kl}(g) M^\mu_{lj}(h^{-1})} dh$

$$= \sum_{\substack{\mu,i,j \\ k,l}} \hat{f}^\mu_{ij} M^\mu_{kl}(g) \underbrace{\int_G M^\mu_{ik}(h) \overline{M^\mu_{jl}}(h)\, dh}_{\frac{1}{n_\mu}\delta_{ij}\delta_{kl}} \quad \text{P-W}$$

$$= \sum_{\mu,i} \frac{\hat{f}^\mu_{ii}}{n_\mu} \chi_\mu(g)$$

$$\Rightarrow f(g) = \sum_{\mu,i} \frac{\hat{f}^\mu_{ii}}{n_\mu} \chi_\mu(g)$$

$\Rightarrow$ $\{\chi_\mu\}$ spans full $L^2(G)^{\text{class}}$.

This helps to find explicit isotypic decomposition of some rep $V \cong \underline{\oplus a_\mu V^{(\mu)}} \equiv \oplus H^{(\mu)}$ $\quad H^{(\mu)} \cong a_\mu V^{(\mu)}$

$$\boxed{\begin{aligned} &\Rightarrow \underline{\chi_V} = \sum_\mu a_\mu \chi_\mu \\ &a_\mu = \langle \chi_\mu, \chi_V \rangle = \int_G \overline{\chi_\mu(g)}\, \chi_V(g)\, dg \end{aligned}}$$

if $V \overset{u}{=} L^2(G)$ of a finite group.

$$\chi_V(e) = \dim V = |G|$$

$$\chi_V(g \neq e) = 0$$

$$a_\mu = \frac{1}{|G|} \sum_g \overline{\chi_\mu(g)}\, \chi_V(g) = \frac{1}{|G|} (\overbrace{\cdot\, n_\mu \cdot |G|}^{g=e} + \overbrace{0}^{g \neq e}) = n_\mu$$

$$|G| = \sum_\mu a_\mu \dim V^\mu = \sum_\mu n_\mu \cdot n_\mu = \sum_\mu n_\mu^2$$

consistent with Peter-Weyl.

because both come from the structure of $L^2(G)$

We can further define class projectors

$$\boxed{\begin{array}{l} \text{Projection onto isotypic subspaces} \\ P^\mu_{ij} := n_\mu \int_G \overline{M^{(\mu)}_{ij}(g)}\, T(g)\, dg \\ P^\mu_{ij} P^\nu_{kl} = \delta_{\mu\nu} \delta_{j,k} P^\nu_{il} \\ T(h) P^\mu_{ij} = \sum_k M^\mu_{ki}(h) P^\mu_{kj} \end{array}}$$

Define $\quad P^\mu := \sum_{i=1}^{n_\mu} P^\mu_{ii}$

$$P_\mu := \sum_{i=1}^{n_\mu} P^{(\mu)}_{ii} = \underline{n_\mu \int_G dg\, \overline{\chi_\mu(g)}\, T(g)}$$

$$P_\mu P_\nu = \sum_{i=1}^{n_\mu} \sum_{j=1}^{n_\nu} P^\mu_{ii} P^\nu_{jj} = \delta_{\mu\nu} \sum_{ij} \delta_{ij} P^\nu_{ij} = \delta_{\mu\nu} P_\nu$$

$$(P_\mu^2 = P_\mu)$$

$$P_\mu^+ = n_\mu \int_G \chi_\mu(g)\, T^+(g)\, dg$$

unitary : $\chi_\mu(g) = \Sigma \lambda_i \quad |\lambda_i| = 1$

$$= n_\mu \int_G \chi_\mu^*(g^{-1})\, T(g^{-1})\, dg$$

$\chi_\mu(g^{-1}) = \Sigma \lambda_i^{-1} = \Sigma \overline{\lambda}_i$

$$= P_\mu$$

$\Rightarrow$ projectors onto isotypic subspaces

$\forall \varphi \in V.$ $\quad T(h) \underbrace{P^\mu \varphi} = T(h) \sum_{i=1}^{n_\mu} P^{(\mu)}_{ii} \varphi = \sum_{ki} M^\mu_{ki}(h) \underbrace{P^{(\mu)}_{ki} \varphi}$

$$P^\mu \varphi \in \mathcal{H}^\mu$$

$$\mathrm{Tr}(P^\mu) = \langle \varphi, P^\mu \varphi \rangle = n_\mu \int_G \underbrace{dg\, \overline{\chi_\mu(g)}\, \chi(g)}_{a_\mu} = n_\mu a_\mu$$

$$= \dim (\mathcal{H}^\mu \cong K^{a_\mu} \otimes V^{(\mu)})$$ dimensions match

8.12.2. Character table of finite groups

For finite groups

we can define a set of class functions

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & \end{cases}$$

where C_i is a distinct conjugacy class.

$\{\delta_{C_i}\}$ is also a basis for the class functions $L^2(G)^{class}$

From above. $\{\chi_\mu\}$ is a basis of $L^2(G)^{class}$ $\Rightarrow$

Theorem. The number of conjugacy classes of a finite group G = the number of irreps.

The character table is an $r \times r$ matrix

E $m_r = |C_r|$

	$m_1 C_1$	$m_2 C_2$	$\cdots$	$m_r C_r$
trivial Γ' V^1	$\chi_1(C_1)$	$\chi_1(C_2)$	$\cdots$	
irreps $\to$ V^2	$\chi_2(C_1)$	$\chi_2(C_2)$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$		$\vdots$
V^r	$\vdots$	$\vdots$		$\chi_r(C_r)$

$$\int_G dg\, \overline{\chi_\mu(g)}\, \chi_\nu(g) = \delta_{\mu\nu} \quad \Longrightarrow$$

$$\frac{1}{|G|} \sum_{\{C_i\}} m_i\, \overline{\chi_\mu(C_i)}\, \chi_\nu(C_i) = \delta_{\mu\nu}$$

define $S_{\mu i} = \sqrt{\frac{m_i}{|G|}}\, \chi_\mu(C_i)$ then

$$\sum_{i=1}^{r} S_{\mu i} S^*_{\nu i} = \delta_{\mu\nu}.$$ S is a unitary matrix

$$\underline{S \cdot S^\dagger = \mathbb{1}_r}$$

There is a dual orthogonality relation

$$\sum_\mu \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i} \delta_{ij}$$

<u>Examples</u>

1. $S_2 \cong \mathbb{Z}_2$

	1	$[(12)]$
1^+	1	1
1^-	1	-1

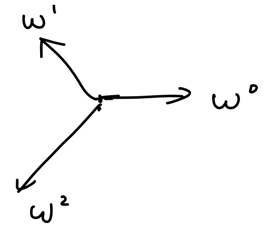
2. $G = \mathbb{Z}_n$ $\quad \#\{C_j\} = n$

$\#\{\text{irreps}\} = n$

$\mathbb{Z}_3$: $\quad \rho_m(\bar{j}) = (\omega_m)^j \qquad \omega_m = e^{i\frac{2\pi}{3}m}$

$\qquad = (\omega_1)^{mj} \qquad \omega = e^{i\frac{2\pi}{3}}$

	$[\bar{0}]$	$[\bar{1}]$	$[\bar{2}]$
ρ_0	1	1	1
ρ_1	1	ω	ω^2
ρ_2	1	ω^2	$\omega^{2\times 2} = \omega$



3. $G = S_3$

σ - 2 cycles $\qquad \tau$ — 3 cycles

$\sigma\tau\sigma = \tau^2 \qquad \tau\sigma\tau^{-1} = \sigma'$

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	A 0	B -1

Given a general rep & a character table. How do we find what irreps it reduces into?

① $\mathbb{R}^3$ rep of S_3:

$$1 = \mathbb{1}_3 \qquad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\chi_V = 3 \qquad 1 \qquad 0$$

$$a_\mu = \langle \chi_\mu , \chi_V \rangle = \int_G (\chi_\mu(g))^* \chi_V(g)\, dg$$

$$= \frac{1}{|G|} \sum_g \overline{\chi_\mu(g)}\, \chi_V(g)$$

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1
V	3	1	0

$$a_{1^+} = \frac{1}{6} (3 + 3 \times 1 + 2 \times 0) = 1$$

$$a_{1^-} = \frac{1}{6} (3 + 3 \times (-1) + 2 \times 0) = 0$$

$$a_2 = \frac{1}{6} ((3 \times 2) + 0 + 0) = 1$$

$$\chi_V = \chi_{1^+} + \chi_2$$

$$V \cong V_{1^+} \oplus V_2$$

② Regular rep of S_3. $\dim(L^2(S_3)) = |S_3| = 6$

$$\chi_V(e) = 6$$

$$\chi_V(g \neq e) = 0$$

$$a_\mu = \langle \chi_\mu , \chi_V \rangle = \frac{1}{|G|} \cdot |G| \cdot \chi_\mu(e) = \underline{\dim V^\mu}$$

$$\boxed{L^2(G) \cong \bigoplus_\mu (\dim V^\mu) \cdot V^\mu}$$