Group Theory And Its Applications

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Why Schur's Lemmas Matter for Physics

In physics we often work with irreducible representations (irreps) of a symmetry group G. For example, angular-momentum multiplets of $SU(2)$, or the momentum eigenstates under translations. Schur's lemmas describe the structure of all linear maps that "respect" the group action. These maps are called *intertwiners* as we have defined earlier. Intertwiners appear constantly in physics, they tell us what kinds of operators commute with all symmetry operations. For instance, in quantum mechanics any operator that commutes with all rotations must act as a scalar on each spin-j multiplet. This is a result of Schur's lemma.

The first lemma says that an intertwiner between two irreps is either zero or an isomorphism. In other words, *irreps do not partially map into one another.* The structure is rigid. The second lemma says that an intertwiner from an irrep to itself must be a scalar multiple of the identity. This explains why degeneracies occur, why Clebsch–Gordan coefficients are unique up to phases, and why symmetry severely restricts operators in quantum systems.

We begin with the first lemma.

8.8.1. Schur's Lemma I & II

Lemma 8.1 (Schur's Lemma I)**.** *Let (T_1, V_1) and (T_2, V_2) be irreducible representations of a group G over a field κ. Let*

$$A : V_1 \longrightarrow V_2$$

be an intertwiner, meaning

$$A\, T_1(g) = T_2(g)\, A \qquad \forall g \in G.$$

Then A is either the zero map or an isomorphism of representations.

Proof. We examine the two subspaces associated with any linear map: its kernel

$$\ker A = \{v \in V_1 \mid A(v) = 0\}$$

and its image

$$\operatorname{im} A = \{A(v) \mid v \in V_1\} \subset V_2.$$

Step 1: $\ker A$ is G-invariant. Take any $v \in \ker A$ and any $g \in G$. Since A intertwines the actions,

$$A(T_1(g)v) = T_2(g)A(v) = T_2(g) \cdot 0 = 0.$$

Thus $T_1(g)v \in \ker A$, so $\ker A$ is an invariant subspace of V_1. Because V_1 is irreducible, its only invariant subspaces are 0 and V_1. Hence either

$$\ker A = 0 \qquad \text{or} \qquad \ker A = V_1.$$

Case 1: $\ker A = V_1$. Then $A(v) = 0$ for all v, so A is the zero map.

Case 2: $\ker A = 0$. Then A is injective.

Step 2: $\operatorname{im} A$ is G-invariant. Let $w = A(v)$ be in $\operatorname{im} A$. Then for any $g \in G$,

$$T_2(g)w = T_2(g)A(v) = A(T_1(g)v).$$

The right-hand side is again in $\operatorname{im} A$. Hence $\operatorname{im} A$ is an invariant subspace of V_2. Since V_2 is irreducible, its only invariant subspaces are 0 and V_2.

Because A is now injective, its image cannot be 0, so

$$\operatorname{im} A = V_2.$$

Thus A is surjective.

Conclusion. A is both injective and surjective, hence an isomorphism. □

The first Schur lemma tells us that intertwiners between two irreps are either zero or isomorphisms. What about *intertwiners from an irrep to itself*? In physics language, these are operators that commute with all symmetry operations and act within a single irreducible multiplet. Schur's second lemma says that, over $\mathbb{C}$, such an operator must be just a scalar multiple of the identity.

Lemma 8.2 (Schur's Lemma II). *Suppose (T, V) is an irreducible representation of a group G on a complex vector space V. Let*

$$A \in \operatorname{Hom}_G(V, V) \quad \textit{(vector space of G-equivariant linear maps $V \to V$)}$$

be an intertwiner, i.e. a linear map $A : V \to V$ such that

$$A\,T(g) = T(g)\,A \qquad \forall g \in G.$$

Then there exists a complex number $\lambda \in \mathbb{C}$ such that

$$Av = \lambda v \qquad \forall v \in V,$$

i.e. $A = \lambda\,\mathbf{1}_V$ is a scalar multiple of the identity.

Proof. Since V is a complex vector space and $\mathbb{C}$ is algebraically closed, a linear operator on V has at least one eigenvalue [1]. Concretely, the characteristic polynomial $\det(x\mathbf{1} - A)$ has a root $\lambda \in \mathbb{C}$, so there exists a nonzero vector $v \in V$ with

$$Av = \lambda v.$$

Let C be the corresponding eigenspace,

$$C = \{w \in V \mid Aw = \lambda w\}.$$

By construction $C \neq 0$. We now show that C is invariant under the action of G. Take any $w \in C$ and any $g \in G$. Using the intertwining property we get

$$A(T(g)w) = T(g)A(w) = T(g)(\lambda w) = \lambda\,T(g)w.$$

Thus $T(g)w$ is again an eigenvector with eigenvalue λ, so $T(g)w \in C$ for all $g \in G$. Hence C is a nonzero G-invariant subspace of V. Because V is irreducible, the only invariant subspaces are 0 and V, so we must have $C = V$. That is, *every* vector in V is an eigenvector of A with the same eigenvalue λ. Therefore $A = \lambda\,\mathbf{1}_V$. $\square$

In physics, the Schur's lemmas describe the precise structure of all operators that commute with a group action, and show how symmetry forces a block decomposition of Hamiltonians and observables.

Remark 8.1 (Intertwiners between decomposed representations). Let V_1 and V_2 be (completely reducible) representations of G. Suppose

$$V_2 \;\cong\; W_1 \oplus W_2$$

is a direct sum decomposition into G–invariant subspaces. Then any intertwiner

$$A : V_1 \longrightarrow V_2, \qquad A\,T_1(g) = T_2(g)\,A,$$

[1] This is only guaranteed in finite dimensions. For a general statement and proof of Schur's lemmas see e.g. Fulton and Harris.

automatically splits as

$$\mathrm{Hom}_G(V_1, V_2) \;\cong\; \mathrm{Hom}_G(V_1, W_1) \;\oplus\; \mathrm{Hom}_G(V_1, W_2).$$

That is, an intertwiner from V_1 into a direct sum must map into each invariant component separately.

Now write the *isotypic decompositions*

$$V_1 \;\cong\; \bigoplus_\nu a_\nu\, V^{(\nu)}, \qquad V_2 \;\cong\; \bigoplus_\nu b_\nu\, V^{(\nu)},$$

where $V^{(\nu)}$ are irreducible representations and a_ν, b_ν their multiplicities. By Schur's Lemma 1,

$$\mathrm{Hom}_G\Big(V^{(\nu)}, V^{(\mu)}\Big) = 0 \qquad (\nu \neq \mu),$$

and by Schur's Lemma 2,

$$\mathrm{Hom}_G\Big(V^{(\nu)}, V^{(\nu)}\Big) \cong \mathbb{C} \cdot \mathrm{Id}.$$

Therefore,

$$\mathrm{Hom}_G(V_1, V_2) \;\cong\; \bigoplus_\nu \mathrm{Hom}(\mathbb{C}^{a_\nu}, \mathbb{C}^{b_\nu}) \;\otimes\; \mathrm{Id}_{V^{(\nu)}}.$$

Conclusion: intertwiners can only mix copies of the same irrep, never mix different irreps. The freedom of an intertwiner lies entirely in its action on the multiplicity (degeneracy) spaces $\mathbb{C}^{a_\nu}$.

Remark 8.2 (Block diagonalisation of symmetry-preserving operators). Let H be a linear operator (Hamiltonian) on a G-representation V such that it commutes with all symmetry operations,

$$[H, T(g)] = 0 \qquad \forall\, g \in G.$$

Then H is an intertwiner, and the previous remark shows that it must take the form

$$H \;\cong\; \bigoplus_\nu (H^{(\nu)} \otimes \mathbf{1}_{V^{(\nu)}}),$$

where each $H^{(\nu)}$ acts only on the multiplicity space $\mathbb{C}^{a_\nu}$ and $\mathbf{1}_{V^{(\nu)}}$ acts on the irrep itself.

Thus the Hilbert space decomposes as

$$\mathcal{H} \;\cong\; \bigoplus_\nu (\mathbb{C}^{a_\nu} \otimes V^{(\nu)}),$$

and in a suitable basis, any symmetry-preserving operator becomes block diagonal:

$$H \;\sim\; \begin{pmatrix} H^{(\nu_1)} & 0 & 0 & \cdots \\ 0 & H^{(\nu_2)} & 0 & \cdots \\ 0 & 0 & H^{(\nu_3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Each block corresponds to a fixed irrep label ν. This is the precise mathematical explanation of the familiar physics idea that "symmetry produces good quantum numbers" and that a Hamiltonian with symmetry decomposes into independent sectors.

Remark 8.3 (Selection rules: commuting vs transforming operators). The block structure above describes all operators O that commute with the group, i.e. $[O, T(g)] = 0$ for all g. In that case O is an intertwiner and has the form

$$O \;\cong\; \bigoplus_{\nu} (O^{(\nu)} \otimes \mathbf{1}_{V^{(\nu)}}),$$

so matrix elements between different irrep sectors vanish

$$\langle \psi_1, O\psi_2 \rangle = 0 \quad \text{if} \quad \psi_1 \in \mathbb{C}^{a_\nu} \otimes V^{(\nu)},\ \psi_2 \in \mathbb{C}^{a_\mu} \otimes V^{(\mu)},\ \nu \neq \mu.$$

This is a first type of selection rule: a symmetry-preserving operator cannot connect states carrying different "good quantum numbers" ν. In many physical situations, however, the interesting operators (dipole moment, spin, current, ...) do *not* commute with G, but they do transform in a controlled way under G. Concretely, suppose O is part of an operator multiplet that transforms as an irrep R of G

$$T(g)\, O\, T(g)^{-1} = D^{(R)}(g)\, O,$$

where $D^{(R)}$ is the matrix representation of g in the irrep R. Then O behaves like a "vector" (or tensor) under G. In this case, symmetry still imposes strong constraints on transition matrix elements

$$\langle \psi_f, O\psi_i \rangle \neq 0 \quad \Longrightarrow \quad \mathrm{Hom}_G(V^{(\nu_f)}, R \otimes V^{(\nu_i)}) \neq 0.$$

This means a nonzero transition is only possible if the irrep carried by the final state appears in the tensor product of the operator irrep R with the irrep of the initial state. For abelian groups this reduces to a simple "charge conservation" rule: the charge of the final state equals the charge of the initial state plus the charge carried by the operator. For example,

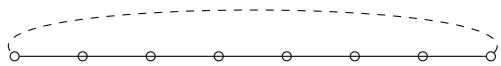
- For a $\mathbb{Z}_2$ parity symmetry, a parity-even operator connects states of the same parity, while a parity-odd operator only connects states of opposite parity.

- For a $U(1)$ symmetry (particle number, spin-z, etc.), an operator with charge q changes the quantum number by q.

For rotational symmetry ($SO(3)$ or $SU(2)$), the same idea underlies the familiar dipole selection rules ($\Delta\ell = \pm 1$, etc.) and is made precise by the Wigner–Eckart theorem. Thus both symmetry-preserving operators and symmetry-transforming operators obey clear selection rules, which are nothing but consequences of Schur's Lemmas and the decomposition of tensor products of irreps. We will return to these ideas in Section. XX.

Example 8.1 (1D tight-binding chain with translation symmetry)**.** Consider a one-dimensional ring of N sites, labelled by $j = 0, 1, \ldots, N-1$. The single-particle Hilbert space is

$$\mathcal{H} = \mathrm{span}\{|j\rangle \mid j = 0, \ldots, N-1\}.$$



The discrete translation operator T acts by shifting the site index:

$$T\,|j\rangle = |j + 1 \bmod N\rangle.$$

The group generated by T is the cyclic group

$$G = C_N = \langle T \mid T^N = \mathbf{1}\rangle.$$

A nearest-neighbour tight-binding Hamiltonian (with hopping amplitude $t > 0$) is

$$\hat{H} = -t \sum_{j=0}^{N-1} \big(|j\rangle\langle j+1| + |j+1\rangle\langle j|\big), \qquad |N\rangle \equiv |0\rangle.$$

By construction $[\hat{H}, T] = 0$, so $\hat{H}$ and T can be diagonalized simultaneously.

Define the Bloch (momentum) eigenstates of T:

$$|k_n\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\mathrm{i}k_n j}\,|j\rangle, \qquad k_n = \frac{2\pi n}{N}, \quad n = 0, 1, \ldots, N-1.$$

Then

$$T\,|k_n\rangle = \frac{1}{\sqrt{N}} \sum_j e^{\mathrm{i}k_n j}\,|j+1\rangle = e^{-\mathrm{i}k_n}|k_n\rangle,$$

so each $|k_n\rangle$ furnishes a one-dimensional irrep of C_N.

Let us now compute the matrix elements of $\hat{H}$ in the $|k_n\rangle$ basis. Acting on a Bloch state, we have

$$\begin{aligned}
\hat{H}|k_n\rangle &= -t \sum_j \big(|j\rangle\langle j+1| + |j+1\rangle\langle j|\big) \frac{1}{\sqrt{N}} \sum_m e^{\mathrm{i}k_n m}|m\rangle \\
&= -t\frac{1}{\sqrt{N}} \sum_j \big(e^{\mathrm{i}k_n(j+1)}|j\rangle + e^{\mathrm{i}k_n j}|j+1\rangle\big) \\
&= -t\frac{1}{\sqrt{N}} \sum_j \big(e^{\mathrm{i}k_n} e^{\mathrm{i}k_n j}|j\rangle + e^{-\mathrm{i}k_n} e^{\mathrm{i}k_n(j+1)}|j+1\rangle\big) \\
&= -t\Big(e^{\mathrm{i}k_n} + e^{-\mathrm{i}k_n}\Big) \frac{1}{\sqrt{N}} \sum_j e^{\mathrm{i}k_n j}|j\rangle \\
&= -2t\cos k_n\,|k_n\rangle.
\end{aligned}$$

Thus the $|k_n\rangle$ are simultaneous eigenstates of T and $\hat{H}$, with eigenvalues

$$T\,|k_n\rangle = e^{\mathrm{i}k_n}|k_n\rangle, \qquad \hat{H}|k_n\rangle = E(k_n)|k_n\rangle, \quad \text{where } E(k_n) = -2t\cos k_n.$$

Equivalently,

$$\langle k_n|\hat{H}|k_m\rangle = -2t\cos k_n\,\delta_{nm}.$$

From the representation-theoretic viewpoint, each momentum k_n labels a one-dimensional irrep (character) of C_N, and the condition $[\hat{H}, T] = 0$ forces $\hat{H}$ to be block diagonal in this irrep basis. In this example each block is 1×1, so the block structure simply says that momentum k is a conserved quantum number for the tight-binding Hamiltonian.

Physically, the momentum quantum numbers arise from characters of the translation group. Mathematically, this is a special case of Pontryagin duality for locally compact Abelian groups. We will not need the full theorem in this course. Anyone interested can refer to Section 11.9 of Moore's note.

8.9. Orthogonality Relations and the Peter-Weyl Theorem

8.9.1. Motivation and Importance

Recall the irreps of $U(1)$. Its elements can be parametrized as

$$z = e^{\mathrm{i}\theta}, \qquad \theta \in [0, 2\pi),$$

with a normalized Haar measure $dg = \frac{d\theta}{2\pi}$. The irreducible unitary representations of $U(1)$ are all one-dimensional and are labeled by integers $n \in \mathbb{Z}$:

$$\rho_n(z) = z^n = e^{\mathrm{i}n\theta}.$$

The matrix element of this one-dimensional irrep is just the function $e^{\mathrm{i}n\theta}$ itself. These functions form an orthonormal basis of $L^2(U(1))$:

$$\langle e^{\mathrm{i}m\theta}, e^{\mathrm{i}n\theta} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-\mathrm{i}m\theta} e^{\mathrm{i}n\theta} = \delta_{mn},$$

and any square-integrable function $f \in L^2(U(1))$ admits the Fourier expansion

$$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{\mathrm{i}n\theta}, \qquad \widehat{f}(n) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, f(\theta) e^{-\mathrm{i}n\theta},$$

which is nothing but the decomposition of $L^2(U(1))$ into its irreducible representations. In this sense, Fourier series provides a simple example for harmonic analysis on a compact group. For a more general compact group G, the space $L^2(G)$ plays the role of the "function space" on which we perform harmonic analysis. The **Peter-Weyl theorem** is the full generalization of this fact to *any* compact group.

Its importance can already be appreciated without knowing any advanced examples. The Peter-Weyl theorem says that matrix elements of irreducible representations play the same role for a general compact group G as the exponentials $e^{\mathrm{i}n\theta}$ do for $U(1)$. In particular:

- **Basis functions on G.** Just as the functions $e^{\mathrm{i}n\theta}$ form a basis for $L^2(U(1))$, Peter-Weyl guarantees that every compact group has its own "Fourier basis" made of representation matrix elements. These functions are orthonormal and span all square-integrable functions on G.

- **Nonabelian Fourier expansion.** Any function $f(g)$ on G can be written as a convergent series of representation matrix elements, completely analogous to the Fourier expansion

  $$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{\mathrm{i}n\theta}.$$

  Peter-Weyl provides the extension of this formula when θ is replaced by a general group element g.

- **Harmonic analysis on general compact groups.** Many problems in physics involve integrating or expanding functions on a compact group (e.g. probability densities, Green functions etc.). Peter-Weyl tells us that all such functions can be built from a universal, orthonormal set of "harmonic" functions determined by the group structure of G.

Thus the Peter-Weyl theorem establishes that every compact group admits a complete Fourier-type analysis, with representation matrix elements playing the role of the basic Fourier modes.

8.9.2. Preliminaries/Review: $L^2(G)$ as a $G \times G$ Representation

Let G be a compact group with normalized Haar measure dg satisfying $\int_G dg = 1$. The Hilbert space

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \int_G |f(g)|^2 \, dg < \infty \}$$

carries a natural unitary representation of $G \times G$:

$$(g_1, g_2) \cdot f(h) = f(g_1^{-1} h g_2).$$

Let (V, T) be a finite-dimensional unitary representation of G. The space of endomorphisms $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ also becomes a unitary representation of $G \times G$ via

$$(g_1, g_2) \cdot S = T(g_1) \, S \, T(g_2)^{-1}.$$

There is a natural $G \times G$-equivariant map

$$\iota : \mathrm{End}(V) \longrightarrow L^2(G), \qquad S \longmapsto \phi_S(g) := \mathrm{Tr}_V(ST(g)^{-1}).$$

For a matrix unit e_{ij} in a basis $\{u_i\}$ of V, the image is the matrix element

$$\phi_{e_{ij}}(g) = M^{(\mu)}_{ij}(g^{-1}),$$

where $M^{(\mu)}(g)$ is the matrix of $T^{(\mu)}(g)$ in that basis.

Extending ι to a direct sum over irreducible representations,

$$\iota : \bigoplus_{\mu} \mathrm{End}(V^{(\mu)}) \;\longrightarrow\; L^2(G),$$

Peter-Weyl theorem states that the image will form an orthogonal basis of $L^2(G)$.

8.9.3. Peter-Weyl Theorem

Theorem 8.3 (Peter-Weyl)**.** *Let G be a compact group. Then the matrix elements of all its irreducible representations form an orthonormal basis of $L^2(G)$. Equivalently,*

$$L^2(G) \;\cong\; \widehat{\bigoplus_{\mu}} \operatorname{End}\big(V^{(\mu)}\big),$$

where the direct sum runs over one representative of each irreducible representation $V^{(\mu)}$ and each appears exactly once.

In the special case $G = U(1)$, this reduces to the familiar Fourier series decomposition above, where the irreducible representations are $z \mapsto z^n$ and the matrix elements are $e^{in\theta}$. To prove this theorem, we follow the following steps:

1. **Finite-dimensionality.** We will show that any irreducible subrepresentation of $L^2(G)$ for a compact group G must be finite-dimensional, thus all irreducible representations of G are finite-dimensional.

2. **Orthogonality.** Using the action of $G \times G$ on $L^2(G)$ and Schur's lemmas, we obtain the orthogonality relations for matrix elements of irreducible unitary representations.

3. **Completeness.** Finally, we show that the span of all matrix coefficients is dense in $L^2(G)$, thus they form an orthonormal basis.

Finite-dimensionality

Proposition 8.4. *Let (T, V) be a unitary irreducible representation of a compact group G on a complex Hilbert space V. Then V is finite dimensional.*

The unitary here can be guaranteed as we know representations of compact groups are unitarizable.

Proof. Fix a nonzero vector $v \in V$. Define a linear operator $L : V \to V$ by

$$L(w) := \int_G \langle T(g)v,\, w\rangle\, T(g)v\, dg, \qquad w \in V,$$

where dg is the Haar measure on G. The map is continuous and G is compact, so the integral is well-defined and L is a bounded operator. L is defined by averaging the "projection onto v" over the group. If V were infinite dimensional, such an average might vanish, but we will show it effectively isolates a finite subspace.

(i) L is an intertwiner. Let $h \in G$ and $w \in V$. Then

$$\begin{aligned} L(T(h)w) &= \int_G \langle T(g)v,\, T(h)w\rangle\, T(g)v\, dg \\ &= \int_G \langle T(h)^\dagger T(g)v,\, w\rangle\, T(g)v\, dg \qquad \text{(unitarity of } T(h)) \\ &= \int_G \langle T(h^{-1}g)\, v,\, w\rangle\, T(g)v\, dg. \end{aligned}$$

Make the change of variables $g = hg'$. By left-invariance of Haar measure, $dg = dg'$, so

$$\begin{aligned} L(T(h)w) &= \int_G \langle T(g')v,\, w\rangle\, T(hg')v\, dg' \\ &= T(h) \int_G \langle T(g')v,\, w\rangle\, T(g')v\, dg' \\ &= T(h)\, L(w). \end{aligned}$$

Thus

$$LT(h) = T(h)L \qquad \forall\, h \in G,$$

so L commutes with the action of G; i.e. L is a G-intertwiner.

(ii) L is a scalar by Schur's lemma. Since (T, V) is irreducible and L is an intertwiner $V \to V$, Schur's lemma implies that

$$L = \lambda\, \mathrm{Id}_V$$

for some scalar $\lambda \in \mathbb{C}$. Compute $\langle v, Lv\rangle$ in two ways. On the one hand,

$$\langle v, Lv\rangle = \int_G \langle T(g)v, v\rangle \overline{\langle T(g)v, v\rangle}\, dg = \int_G \left|\langle v, T(g)v\rangle\right|^2 dg.$$

The integrand is a continuous, nonnegative, bounded function of g. On the other hand, $L = \lambda\, \mathrm{Id}_V$ implies

$$\langle v, Lv\rangle = \lambda\, \|v\|^2.$$

At $g = e$ we have $\langle v, T(e)v\rangle = \langle v, v\rangle = \|v\|^2 > 0$, so by continuity there is a neighborhood of e on which $|\langle v, T(g)v\rangle|^2$ is bounded below by some $\delta > 0$. Hence the integral is strictly positive:

$$\langle v, Lv\rangle > 0,$$

i.e. $\lambda > 0$.

(iii) Trace of L and finiteness of $\dim V$**.** Because $T(g)$ is unitary, traces $\|T(g)v\|^2 = \|v\|^2$. Using linearity of the trace and Fubini-type interchange of integral and trace (justified here because the integrand is norm-bounded and G is compact), we obtain

$$\mathrm{Tr}(L) = \sum_i \langle v_i, L(v_i)\rangle = \sum_i \int_G \langle v_i, T(g)v\rangle \langle T(g)v, v_i\rangle\, dg = \sum_i \int_G \left|\langle v_i, T(g)v\rangle\right|^2 dg$$

$$= \int_G \sum_i \left|\langle v_i, T(g)v\rangle\right|^2 dg = \int_G \|T(g)v\|^2 \, dg = \|v\|^2 \operatorname{vol}(G),$$

which is finite. But since $L = \lambda \operatorname{Id}_V$, its trace is also

$$\operatorname{Tr}(L) = \lambda \dim V.$$

We have just shown that $\operatorname{Tr}(L)$ is finite and that $\lambda > 0$, hence $\dim V$ must be finite

$$\dim V = \frac{\operatorname{Tr}(L)}{\lambda} < \infty.$$

□

Orthogonality Relations for Matrix Elements of Irreps

Let G be a compact group, and equip $L^2(G)$ with the Hermitian inner product

$$\langle \Psi_1, \Psi_2 \rangle := \int_G \overline{\Psi_1(g)}\, \Psi_2(g)\, dg,$$

where we normalize the Haar measure so that $\operatorname{vol}(G) = 1$.

Let $\{V^{(\mu)}\}$ be a set of representatives of the distinct isomorphism classes of irreducible unitary representations of G. From Step 1 we know that each irreducible representation is finite-dimensional. For each $V^{(\mu)}$ choose an orthonormal basis

$$w_i^{(\mu)}, \qquad i = 1, \ldots, n_\mu,$$

where $n_\mu = \dim_{\mathbb{C}} V^{(\mu)}$. For each irreducible representation $T^{(\mu)} : G \to U(n_\mu)$ define the matrix elements $M_{ij}^{(\mu)}(g)$ by

$$T^{(\mu)}(g)\, w_i^{(\mu)} = \sum_{j=1}^{n_\mu} M_{ji}^{(\mu)}(g)\, w_j^{(\mu)}.$$

Let $A : V^{(\mu)} \to V^{(\nu)}$ be any linear map. Define the averaged map

$$\widetilde{A} := \int_G T^{(\nu)}(g)\, A\, T^{(\mu)}(g^{-1})\, dg.$$

A direct computation shows that $\widetilde{A}$ is an intertwiner. For any $h \in G$,

$$\begin{aligned} T^{(\nu)}(h)\, \widetilde{A} &= \int_G T^{(\nu)}(hg)\, A\, T^{(\mu)}(g^{-1})\, dg \\ &= \int_G T^{(\nu)}(g)\, A\, T^{(\mu)}(g^{-1}h)\, dg \qquad \text{(left-invariance of Haar measure)} \\ &= \widetilde{A}\, T^{(\mu)}(h). \end{aligned}$$

Thus $\widetilde{A}$ is a G-intertwiner. By Schur's lemma,

$$\widetilde{A} = \delta_{\mu,\nu}\, \widehat{A},$$

where $\widehat{A} = c_A \mathrm{Id}_{V^{(\mu)}}$ is a scalar multiple of the identity on $V^{(\mu)}$. On the other hand, write $\widetilde{A}$ in components with respect to the orthonormal bases $\{w_i^{(\mu)}\}$ and $\{w_i^{(\nu)}\}$. From its definition, we have

$$[\widetilde{A}]_{ia}^{(\nu\mu)} = \sum_{l,\alpha} \int_G M_{il}^{(\nu)}(g)\, A_{l\alpha}\, M_{\alpha a}^{(\mu)}(g^{-1})\, dg \equiv \delta_{\mu,\nu} c_A \delta_{i,a}. \qquad (*)$$

To determine c_A, set $\mu = \nu$ and take the trace (sum over $i = a$):

$$\begin{aligned} n_\mu c_A &= \sum_i [\widetilde{A}]_{ii}^{(\mu\mu)} \\ &= \sum_{i,l,\alpha} \int_G M_{il}^{(\mu)}(g)\, A_{l\alpha}\, M_{\alpha i}^{(\mu)}(g^{-1})\, dg \\ &= \int_G \mathrm{Tr}\big(M^{(\mu)}(g)\, A\, M^{(\mu)}(g^{-1})\big)\, dg \\ &= \int_G \mathrm{Tr}(A)\, dg \\ &= \mathrm{Tr}(A), \end{aligned}$$

therefore

$$c_A = \frac{1}{n_\mu} \mathrm{Tr}(A).$$

Now choose A to be the matrix unit e_{jk} (so $\mathrm{Tr}(e_{jk}) = \delta_{jk}$). Plugging this into (*) gives

$$\sum_{l,\alpha} \int_G M_{il}^{(\mu)}(g)\, (e_{jk})_{l\alpha}\, M_{\alpha a}^{(\mu)}(g^{-1})\, dg = \frac{1}{n_\mu}\, \delta_{jk}\, \delta_{ia}.$$

Since $(e_{jk})_{l\alpha} = \delta_{jl}\, \delta_{k\alpha}$, this simplifies to

$$\int_G M_{ij}^{(\mu)}(g)\, M_{ka}^{(\mu)}(g^{-1})\, dg = \frac{1}{n_\mu}\, \delta_{jk}\, \delta_{ia}.$$

Finally, using unitarity $M^{(\mu)}(g^{-1}) = M^{(\mu)}(g)^\dagger$, $M_{ka}^{(\mu)}(g^{-1}) = \overline{M_{ak}^{(\mu)}(g)}$ and renaming indices, we obtain the Schur orthogonality relations

$$\boxed{\int_G \overline{M_{ij}^{(\mu)}(g)}\, M_{k\ell}^{(\nu)}(g)\, dg = \frac{1}{n_\mu}\, \delta_{\mu\nu}\, \delta_{ik}\, \delta_{j\ell}.}$$

It is sometimes convenient to define

$$\phi_{ij}^{(\mu)}(g) := \sqrt{n_\mu}\, M_{ij}^{(\mu)}(g),$$

so that

$$\int_G \overline{\phi_{ij}^{(\mu)}(g)}\, \phi_{k\ell}^{(\nu)}(g)\, dg = \delta_{\mu,\nu}\, \delta_{ik}\, \delta_{j\ell}.$$

Thus the family $\{\phi_{ij}^{(\mu)}\}_{\mu,i,j}$ is an orthonormal set in $L^2(G)$.

Completeness of Matrix Elements

We now prove that $\{\phi_{ij}^{(\mu)}\}_{\mu,i,j}$ form an *orthonormal basis* of $L^2(G)$ by showing they are complete.

Let

$$W := \overline{\mathrm{span}\{\phi_{ij}^{\mu}\}} \subset L^2(G)$$

be the closed span of all matrix coefficients, and let $\mathcal{N} := W^{\perp}$ denote its orthogonal complement. We must show that $\mathcal{N} = \{0\}$.

Recall the right regular representation

$$(R(h)f)(g) = f(gh).$$

Because Haar measure is right-invariant, $R(h)$ is unitary on $L^2(G)$. Since each ϕ_{ij}^{μ} transforms among functions of the same irrep,

$$R(h)\phi_{ij}^{\mu}(g) = \phi_{ij}^{\mu}(gh) = \sum_k M_{kj}^{(\mu)}(h)\,\phi_{ik}^{\mu}(g),$$

the space W is $R(G)$-invariant. Hence its orthogonal complement $\mathcal{N} = W^{\perp}$ is also $R(G)$-invariant. Thus $\mathcal{N}$ is a *unitary subrepresentation* of G [2]. Now, if $\mathcal{N} \neq \{0\}$, it is either finite-dimensional or infinite-dimensional. If finite, then it is completely reducible; if infinite, we have just proved that there is no infinite-dimensional irrep, hence $\mathcal{N}$ is reducible. Suppose for the moment that $\mathcal{N}$ is decomposable as a direct sum of irreducible unitary representations (see Remark below). Then there exists at least one irreducible subrepresentation

$$(T^{(\mu)}, V^{(\mu)}) \subset \mathcal{N}.$$

Choose an orthonormal basis $\{f_j\}_{j=1}^{n_\mu}$ of $V^{(\mu)}$. By definition, the right regular representation restricted to $V^{(\mu)}$ acts as $T^{(\mu)}$, so for each $g \in G$ we can write

$$R(g)\,f_j = \sum_{k=1}^{n_\mu} M_{kj}^{(\mu)}(g)\,f_k, \tag{8.1}$$

where $M_{kj}^{(\mu)}(g)$ are the matrix elements of $T^{(\mu)}(g)$ in the basis $\{f_j\}$. Evaluating (8.1) at a point

[2] Same proof as proposition in Sec. 8.6. Take $f \in \mathcal{N}$, $w \in W$, and $g \in G$. Then

$$\langle R(g)f, w\rangle = \langle f, R(g^{-1})w\rangle = 0,$$

since W is invariant and $R(g^{-1})w \in W$. Hence $R(g)f \in W^{\perp}$. Therefore

$$R(g)\mathcal{N} \subset \mathcal{N} \quad \forall g,$$

i.e. $\mathcal{N}$ is invariant.

$h \in G$ gives

$$f_j(hg) = \sum_{k=1}^{n_\mu} M_{kj}^{(\mu)}(g)\, f_k(h). \tag{8.2}$$

Now set $h = e$ (the identity element). Since equality in (8.2) holds as functions on G, we obtain

$$f_j(\cdot) = \sum_{k=1}^{n_\mu} f_k(e)\, M_{kj}^{(\mu)}(\cdot). \tag{8.3}$$

Thus each f_j is a linear combination of the matrix elements $M_{kj}^{(\mu)}$, so $f_j \in W$ for all j.

But by construction $f_j \in V^{(\mu)} \subset \mathcal{N} = W^\perp$, so the only way (8.3) can hold is if $f_j = 0$ for all j, i.e. $V^{(\mu)} = \{0\}$. This contradicts our assumption that $V^{(\mu)}$ is a non-zero irreducible subrepresentation. Hence we must have

$$\mathcal{N} = W^\perp = \{0\},$$

and therefore

$$W = \overline{\operatorname{span}\{\phi_{ij}^{(\mu)}\}} = L^2(G).$$

We conclude that the functions $\phi_{ij}^{(\mu)}$ form an orthonormal basis of $L^2(G)$.

Remark 8.4. In the argument above we assumed, following Moore, that the orthogonal complement $\mathcal{N} = W^\perp$ is a direct sum of finite-dimensional irreducible subrepresentations. This allows us to pick a nonzero finite-dimensional irreducible subrepresentation $V^{(\mu)} \subset \mathcal{N}$ and obtain the contradiction which forces $\mathcal{N} = \{0\}$.

However, this assumption is not automatic. Earlier we have shown that finite-dimensional unitary representations are completely reducible, but in infinite dimensions one must rule out the possibility of *reducible but indecomposable* unitary representations. In fact, one can show that *any unitary representation of a compact group is completely reducible.* A proof can be found in standard references such as A.W. Knapp, *Lie Groups Beyond an Introduction*, Chap. 4.3.

Remark 8.5. The Peter-Weyl theorem establishes that the matrix elements $M_{ij}^{(\mu)}(g)$ form an orthogonal basis for $L^2(G)$. We can rephrase this result in a powerful, coordinate-free language that avoids the arbitrary choice of basis vectors $\{w_i\}$. This goes back to what we discussed in Sec. 8.9.2 and regular representations. The Peter-Weyl theorem can thus be summarized as an isomorphism of $G \times G$ representations

$$L^2(G) \cong \bigoplus_\mu \operatorname{End}(V^{(\mu)}), \tag{8.4}$$

where the RHS sums over all distinct irreps.

For finite groups, we have the following immediate corollary just by counting the dimensions.

Corollary 8.5. *Let G be a finite group. Then the sum of the squares of the dimensions of all its irreducible representations equals the order of the group:*

$$|G| = \sum_{\mu} (\dim V^{(\mu)})^2.$$

In the following we illustrate the Peter-Weyl theorem on a few finite and compact groups.

Example 8.2. $G = \mathbb{Z}_2$

Let $G = \mathbb{Z}_2 = \{1, \sigma\}$ with $\sigma^2 = 1$. Then

$$L^2(\mathbb{Z}_2) \cong \mathbb{C}^2$$

with inner product

$$\langle \psi, \varphi \rangle = \frac{1}{2}(\overline{\psi(1)}\varphi(1) + \overline{\psi(\sigma)}\varphi(\sigma)).$$

There are precisely two irreps, both one-dimensional:

$$\rho_+(1) = 1, \quad \rho_+(\sigma) = 1, \qquad \text{(trivial)},$$
$$\rho_-(1) = 1, \quad \rho_-(\sigma) = -1, \qquad \text{(sign)}.$$

Given any function $\varphi \in L^2(\mathbb{Z}_2)$, write coefficients

$$\varphi_+ := \frac{\varphi(1) + \varphi(\sigma)}{2}, \qquad \varphi_- := \frac{\varphi(1) - \varphi(\sigma)}{2}.$$

Then

$$\varphi(g) = \varphi_+\,\rho_+(g) + \varphi_-\,\rho_-(g),$$

so $\{\rho_+, \rho_-\}$ is an orthonormal basis of $L^2(\mathbb{Z}_2)$.

One can construct projection operators

$$P_\pm := \frac{1}{2}(I \pm T(\sigma)),$$

where $T(\sigma)$ is a faithful representation operator on a representation space V. One checks

$$P_\pm^2 = P_\pm, \qquad P_+P_- = 0, \qquad P_+ + P_- = I,$$

and $P_\pm$ project V onto the subspaces spanned by $\rho_\pm$, respectively. This is the simplest nontrivial example of Peter-Weyl.

Example 8.3. $\mathbb{Z}_n$ with $n > 2$

Let $G = \mathbb{Z}_n = \{\omega^j \mid j = 0, 1, \ldots, n-1\}$ where

$$\omega := e^{2\pi i/n}$$

is a primitive nth root of unity.

Since G is abelian, *all* irreducible representations are one-dimensional and can be simultaneously diagonalized. The set of irreps is naturally indexed by integers $m \in \mathbb{Z}$ modulo n:

$$(\rho_m, V_m), \qquad V_m \cong \mathbb{C},$$

where

$$\rho_m(\omega^j) = \omega^{mj} = e^{2\pi i m j/n}.$$

The Peter-Weyl orthogonality relation becomes the discrete Fourier orthogonality:

$$\frac{1}{n} \sum_{g \in G} \overline{\rho^{(m_1)}(g)}\, \rho^{(m_2)}(g) = \delta_{m_1 - m_2 \equiv 0 \ (\mathrm{mod}\ n)}.$$

Every function $\Psi : \mathbb{Z}_n \to \mathbb{C}$ can be expanded uniquely as a finite Fourier series:

$$\Psi(g) = \sum_{m=0}^{n-1} \widehat{\Psi}_m\, \rho^{(m)}(g),$$

where the Fourier coefficients are

$$\widehat{\Psi}_m = \frac{1}{|G|} \sum_{g \in G} \overline{\rho^{(m)}(g)}\, \Psi(g) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-2\pi i m j/n}\, \Psi(\omega^j).$$

Example 8.4. $G = U(1)$: we have seen this before

Let $G = U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ with Haar measure $d\theta/2\pi$ on $[0, 2\pi)$. For each integer $n \in \mathbb{Z}$ define the one–dimensional unitary representation

$$\rho_n(e^{i\theta}) := e^{in\theta}.$$

Each ρ_n acts on $V_n \cong \mathbb{C}$.

The inner product on $L^2(U(1))$ is

$$\langle f, g \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi}\, f(\theta)\, \overline{g(\theta)}.$$

A straightforward computation gives the orthogonality relation

$$\langle \rho_n, \rho_m \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{in\theta}\, e^{-im\theta} = \delta_{nm}.$$

Thus the characters ρ_n form an orthonormal set.

Peter-Weyl now says that $\{\rho_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2(U(1))$. In other words, every square-integrable function ψ on the circle admits a Fourier expansion

$$\psi(\theta) = \sum_{n\in\mathbb{Z}} a_n\, e^{in\theta}, \qquad a_n = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-in\theta}\, \psi(\theta),$$

with convergence in the L^2 sense. This is nothing but the usual Fourier series, now interpreted as the harmonic analysis of the compact group $U(1)$.

Example 8.5. $G = S_3$

S_3 has six elements and three irreps, as we have seen earlier:

- the *trivial* one-dimensional irrep V_{triv},
- the *sign* one-dimensional irrep V_{sgn},
- the *standard* two-dimensional irrep V_2.

The dimensions satisfy

$$|S_3| = 6 = 1^2 + 1^2 + 2^2,$$

in agreement with the general identity $|G| = \sum_\mu (\dim V^{(\mu)})^2$ for finite groups.

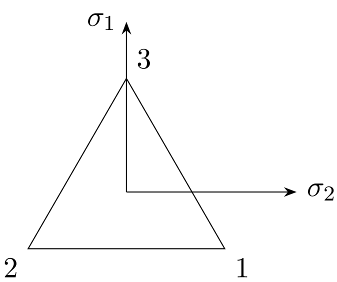
The left regular representation of S_3 on $L^2(S_3) \cong \mathbb{C}^6$ therefore decomposes as

$$L^2(S_3) \;\cong\; V_{\text{triv}} \oplus V_{\text{sgn}} \oplus 2V_2,$$

i.e. we get one copy of the trivial irrep, one copy of the sign irrep, and two copies of the standard irrep. For the one-dimensional irreps we have

$$\rho_{\text{triv}}(g) = 1 \qquad \forall g \in S_3,$$
$$\rho_{\text{sgn}}(g) = \begin{cases} 1, & g \text{ even permutation,} \\ -1, & g \text{ odd permutation.} \end{cases}$$

The standard two-dimensional irrep can be realized as the action of S_3 on the plane containing an equilateral triangle.



The above basis choice gives the following matrices:

Permutation	$M^{(2)}(g)$
e	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
(12)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
(123)	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
(132)	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
(13)	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
(23)	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Here the basis vectors σ_1, σ_2 span the plane orthogonal to the invariant vector $(1,1,1)$, which carries the standard irrep of S_3.

Peter-Weyl orthogonality check. For a finite group, the Haar measure is just the uniform measure, so the inner product on $L^2(S_3)$ is

$$\langle f, g \rangle = \frac{1}{|S_3|} \sum_{g \in S_3} f(g)\, \overline{g(g)}.$$

Let $M_{ij}^{(\mu)}(g)$ denote the (i,j) matrix element of the irrep $M^{(\mu)}(g)$ in a fixed orthonormal basis of $V^{(\mu)}$. The Peter-Weyl orthogonality relation specialised to S_3 says

$$\big\langle M_{ij}^{(\mu)}, M_{k\ell}^{(\nu)} \big\rangle = \frac{1}{n_\mu}\, \delta_{\mu\nu}\, \delta_{ik}\, \delta_{j\ell}, \qquad n_\mu = \dim V^{(\mu)}.$$

Let us check a few cases explicitly.

(a) Trivial vs. sign. Since $\rho_{\text{triv}}(g)\, \overline{\rho_{\text{sgn}}(g)}$ is $+1$ on even permutations and -1 on odd permutations, and there are three of each, we get

$$\big\langle \rho_{\text{triv}}, \rho_{\text{sgn}} \big\rangle = \frac{1}{6} \sum_{g \in S_3} \rho_{\text{triv}}(g)\, \overline{\rho_{\text{sgn}}(g)} = \frac{1}{6}\,(3-3) = 0,$$

as required.

(b) Trivial vs. a diagonal matrix element of V_2. Consider $M_{11}^{(2)}$, the $(1,1)$ entry of the standard irrep. One finds that

$$\langle \rho_{\text{triv}}, M_{11}^{(2)} \rangle = \frac{1}{6} \sum_{g \in S_3} M_{11}^{(2)}(g) = \frac{1}{6}\,(1 + 1 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}) = 0.$$

(c) Self–inner product of a matrix element. A direct computation shows

$$\langle M_{11}^{(2)}, M_{11}^{(2)} \rangle = \frac{1}{6} \sum_{g \in S_3} |M_{11}^{(2)}(g)|^2 = \frac{1}{2} = \frac{1}{n_2},$$

which agrees with the general formula with $n_2 = 2$.

Example 8.6. $G = S_4$

Finally we briefly sketch the situation for $G = S_4$. It has $|S_4| = 24$ elements, and we know that similar to S_3, it has the trivial, sign, and standard irreps, each of dimensions 1, 1, and 3, respectively. The characters of each conjugacy class for V_{triv} and V_{sgn} are immediately known, and those of V_{std} can be easily computed using the basis $\{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$.

	1 [(1)]	6 [(12)]	3 [(12)(34)]	8 [(123)]	6 [(1234)]
V_{triv}	1	1	1	1	1
V_{sgn}	1	-1	1	1	-1
V_2	2	0	2	-1	0
V_{std}	3	1	-1	0	-1
$V_{\text{sgn}\otimes\text{std}}$	3	-1	-1	0	1

But $24 - 1 - 1 - 3^2 = 13$, so there must be more irreps to account for the full dimension of the regular representation. We try to find the remaining irreps of S_4.

The $V_{\text{sgn}\otimes\text{std}}$ irrep. Recall that we can construct new representations using tensor products of representations. The dimension of the new representation is the product of the dimensions of the original representations. What if we tensor product V_{sgn} and V_{std}? This gives a new three-dimensional representation $V_{\text{sgn}\otimes\text{std}}$ with characters $\chi_{\text{sgn}\otimes\text{std}} = \chi_{\text{sgn}}\chi_{\text{std}}$, as shown in the last row of the table. Is it an irrep? Yes.

Proof. Let ρ be any irreducible representation of a group G and let σ be a one-dimensional representation. The group action of $\sigma \otimes \rho$ on the same underlying vector space is given by

$$(\sigma \otimes \rho)(g)v = \sigma(g)\,\rho(g)v.$$

Then a subspace $W \subset V$ is invariant under $\sigma \otimes \rho$ if and only if it is invariant under ρ, since $\sigma(g)$ is just a nonzero scalar. Thus $\sigma \otimes \rho$ is irreducible if and only if ρ is irreducible. □

The remaining V_2 irrep. We have so far accounted for $1^2 + 1^2 + 3^2 + 3^2 = 20$ dimensions, so there are four dimensions left to fill. The only possibilities are a single two-dimensional irrep V_2, or four one-dimensional irreps. We know the latter is impossible, because there can be only two one-dimensional irreps of S_4 (trivial and sign).

Proof. A one-dimensional representation of S_4 is a group homomorphism

$$\chi : S_4 \to \mathbb{C}^\times.$$

Assume conjugacy classes have the characters/matrix elements

$$\chi((12)) = a, \qquad \chi((123)) = b, \qquad \chi((1234)) = c,$$

then they have to satisfy

$$\begin{aligned} (12)^2 = e &\implies a^2 = 1, \\ (123)^3 = e &\implies b^3 = 1, \\ (1234)^4 = e &\implies c^4 = 1. \end{aligned}$$

As $(12)(23) = (231) = (123)$, we have the relation $b = \chi((123)) = \chi((12))\,\chi((23)) = a^2 = 1$. Similarly, from $(1234)(12) = (134)$, we obtain $c\,a = 1$. The only free parameter left is $a = \chi((12))$, and $a^2 = 1$ forces $a = \pm 1$. Thus there are precisely two one-dimensional representations:

$$\chi_{\text{triv}}(g) = 1, \qquad \chi_{\text{sgn}}(g) = \text{sgn}(g).$$

$\square$

Proof. (1D reps = abelianization characters) Again, we use the fact that one-dimensional representation of a group G is a homomorphism

$$\chi : G \to \mathbb{C}^\times.$$

Since $\mathbb{C}^\times$ is abelian, every such χ kills all commutators: for any $x, y \in G$,

$$\chi([x, y]) = \chi(xyx^{-1}y^{-1}) = \chi(x)\chi(y)\chi(x)^{-1}\chi(y)^{-1} = 1.$$

Thus every one-dimensional representation factors through the quotient

$$G \longrightarrow G/[G, G],$$

the *abelianization* of G. Therefore the number of one-dimensional representations of G equals the number of homomorphisms from $G/[G, G]$ to $\mathbb{C}^\times$.

Since $[G, G]$ is contained in the kernel of any homomorphism into an abelian group (see Problem 13 of HW04),

$$[S_4, S_4] \subseteq A_4,$$

as A_4 is the kernel of the sign map. Conversely, A_4 is generated by 3-cycles, and every 3-cycle is a commutator in S_4, e.g., $(132) = (12)(23)(12)^{-1}(23)^{-1}$. Therefore, $[S_4, S_4] = A_4$. The abelianization of S_4 is thus

$$S_4/[S_4, S_4] \;\cong\; S_4/A_4 \;\cong\; \mathbb{Z}_2.$$

Therefore one-dimensional representations of S_4 are precisely the group homomorphisms from $\mathbb{Z}_2$ to $\mathbb{C}^\times$, which are the trivial and the sign representations of S_4. $\square$

Finally, the isotypic decomposition of the left regular representation of S_4 is given as

$$L^2(S_4) \cong V_{\text{triv}} \oplus V_{\text{sgn}} \oplus V_2 \oplus V_{\text{std}} \oplus V_{\text{sgn}\otimes\text{std}}.$$

We can work out the characters of V_2 by matching the characters between both sides, thereby completing the character table above. In the next lectures we will learn systematic ways to construct irreps for general groups.