Recap:

1. To study representation of groups. we start with the "biggest" and most natural one: the regular representation $L^2(G)$.

   To define $L^2(G)$, we need G to be

   1. locally compact $\Rightarrow$ Haar measure. i.e. meaningful integration

   2. compact $\Rightarrow$ finite integration ($\because$ bounded)

   $\hookrightarrow$ this leads to well-defined inner product

   $$\int_G |f(g)|^2 d\mu(g) < \infty .$$

   Natural $G \times G$ action : $(g_1, g_2) \longrightarrow L(g_1) R(g_2^{-1})$

   $$(L_h f)(g) = f(h^{-1} g)$$
   $$(R_h f)(g) = f(gh)$$

   "biggest": for any (T, V)

   $$i : \mathrm{End}(V) \longrightarrow L^2(G)$$
   $$S \longmapsto f_S = \mathrm{Tr}_V (S T(g^{-1}))$$

   $$f_{e_{ij}} = [M(g^{-1})]_{ji} \overset{V \to V^*}{=} [M^*(g)]_{ij} \quad \text{(dual rep.)}$$

2. We don't need to study all possible reps of a group, but only the unique "building blocks" all of which are contained in $L^2(G)$. This leads to the need of "reducing" $L^2(G)$, i.e. finding subspaces invariant under group actions.

<u>Reducible</u> vs <u>irreducible</u>

↳ proper, nontrivial invariant subspaces

completely reducible: $V \cong W_1 \oplus W_2 \oplus \cdots$ all invariant

indecomposable $V \cong W_1 \oplus W_1^{\perp}$ W_1 invariant, $W_1^{\perp}$ not

8.7 Reducible & irreducible representations (cont.)

Examples

1. $G = \mathbb{Z}_2$ 1-D rep $V = \mathbb{R}$

   trivial : $\rho_+(1) = \rho_+(-1) = 1$

   $\rho_-(1) = 1, \quad \rho_-(-1) = -1$

2. $G = \mathbb{Z}_2 \cong S_2 = \{e, \tau\}$

   $M(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   $M(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   $A = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow \tilde{M}(\tau) = A^{-1} M A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

   $\rho_+(e) = \rho_+(\tau) = 1$

   $\rho_-(e) = 1, \quad \rho_-(\tau) = -1$

   $(T, V) \cong \rho_+ \oplus \rho_-$ completely reducible

3. $G = U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$ $V = \mathbb{C}$.

   $\rho_n(z) = z^n$ for $\forall n \in \mathbb{Z}$. Q: why integer?

   $\rho_n(z_1 z_2) = (z_1 z_2)^n = \rho_n(z_1)\, \rho_n(z_2)$ $e^{i\alpha(\theta_1+\theta_2)}$?

   <u>are there other irreps ?</u>

4. Finite-dimensional representations of Abelian groups are completely reducible.

Choosing an ordered orthonormal (ON) basis. s.t. all $M(g)$ ($\forall g \in G$) are commuting unitary matrices. over the complex field

$$M(g_i)M(g_j) = M(g_j)M(g_i) \qquad \forall g_i, g_j \in G$$

as required by the abelian property.

$\Rightarrow$ M's can be simultaneously diagonized (spectral theorem)

$$M(z) = \mathrm{diag}\{\lambda_1(z), \lambda_2(z), \cdots \lambda_d(z)\}$$

For $G = U(1)$. any f.d. rep on $V \cong \mathbb{C}^d$

$$M(z) = \mathrm{diag}\{\rho_{n_1}(z), \rho_{n_2}(z), \cdots \rho_{n_d}(z)\}$$

$$V \cong \rho_{n_1} \oplus \rho_{n_2} \oplus \cdots \oplus \rho_{n_d}.$$

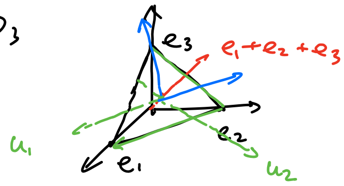
Finite. compact Abelian groups all irreps are 1D in $\mathbb{C}$

domain dependence: e.g. $SO(2)$ $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\to \left(\begin{array}{c|c} e^{i\theta} & 0 \\ \hline 0 & e^{-i\theta} \end{array}\right)$$

So reducible on $\mathbb{C}$. but irreducible on $\mathbb{R}$.

5. Non abelian $S_3 \cong D_3$



canonical on $\mathbb{R}^3 = \mathrm{span}\{e_1, e_2, e_3\}$

$T(\sigma)e_i = e_{\sigma(i)}$

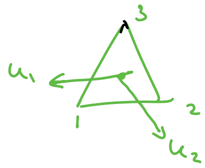
① $u_0 = e_1 + e_2 + e_3$ invariant subspace W

$T(\sigma)u_0 = u_0 \Rightarrow T|_W = 1_W$. trivial rep.

② its complement $W^\perp = \mathrm{span}\{u_1, u_2\}$

a. $u_1 = e_1 - e_2$

$u_2 = e_2 - e_3$



$\sum_j M_{ji} u_j = u_i$

$T((12)) \cdot u_1 = -u_1$

$T((12)) \cdot u_2 = u_1 + u_2$

$M((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$T((23))u_1 = u_1 + u_2$

$T((23))u_2 = -u_2$

$M((23)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

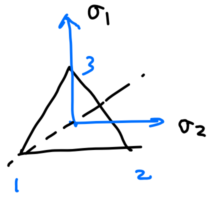
$T((13))u_1 = -u_2$

$T((13))u_2 = -u_1$

$M((13)) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

unitary rep. not unitary mat.

b. using ON basis.



$M((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$T[(23)]\sigma_1 = -\frac{1}{2}\sigma_1 + \frac{\sqrt{3}}{2}\sigma_2$

$T[(23)]\sigma_2 = \frac{\sqrt{3}}{2}\sigma_1 + \frac{1}{2}\sigma_2$

$M[(23)] = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

similarly.

$M[(13)] = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ $\quad M[(123)] = R(\frac{2}{3}\pi) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$\mathbb{R}^3 \cong W \oplus W^\perp$

6. more generally. consider rep. of S_n on $\mathbb{R}^n$

$u_0 = \sum e_i$ invariant subspace W

$W = \{ x \sum e_i \ , x \in \mathbb{R} \}$

$W^\perp = \{ \sum x_i e_i \mid \sum x_i = 0 \ , x_i \in \mathbb{R} \}$

$\left(\sum_{ij} x_i \langle e_i , e_j \rangle = \sum_{ij} x_i \delta_{ij} = \sum_j x_j = 0 \right)$

$\boxed{\text{Both } W \text{ and } W^\perp \text{ are irreducible.}}$

S_3: $W^\perp = \text{span} \{ e_1 - e_2 , e_2 - e_3 \}$

We saw it is an irrep on $\mathbb{R}^2$ in lecture

is $W^\perp$ irrep in general ?

consider $u = \sum x_i e_i \in U$. $U \subset W^\perp$ an invariant subspace

not all x_i equal. otherwise $\sum x_i = 0 \Rightarrow x_i = 0$

WLOG. assumes $x_1 \neq x_2$

$u - \sigma_{12} u = x_1 e_1 + x_2 e_2 - x_1 e_2 - x_2 e_1$

$= (x_1 - x_2)(e_1 - e_2) \in U$ ($\because U$ invariant)

$\Rightarrow e_1 - e_2 \in U$

$\Rightarrow$ All $\tau \in S_n$ acts on $e_1 - e_2$

$(123)(e_1 - e_2) = e_2 - e_3 \in U$. etc.

$\Rightarrow \dim \text{span} \{ e_1 - e_2 , e_2 - e_3 , \cdots e_{n-1} - e_n \} = n-1$

$U \cong W^\perp$

7. Above examples are <u>completely reducible</u>.

Now consider example

a. $u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ on $K^2 = \mathbb{R}^2, \mathbb{C}^2$

$\left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right\}$ is an invariant subspace.

b. $A \in GL(n, K)$

$$\det : A \longmapsto \det A \overset{\log}{\longmapsto} \log|\det A|$$

$$A \cdot B \longmapsto \det AB = \det A \cdot \det B$$

$$\longmapsto \log|\det A| + \log|\det B|$$

$$A \longrightarrow \begin{pmatrix} 1 & \log|\det A| \\ 0 & 1 \end{pmatrix}$$

$$T(A)\, T(B) = \begin{pmatrix} 1 & \log|\det A| + \log|\det B| \\ 0 & 1 \end{pmatrix} = T(AB)$$

8. Semidirect product $H \rtimes_\alpha G$

[recall direct product. $H \times G$

$$(h_1, g_1)(h_2, g_2) = (h_1 h_2, g_1 g_2)$$

semidirect product has an additional

G-action of G on H:

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \alpha_{g_1}(h_2), g_1 g_2)$$

or, direct product is semidirect product

with a trivial action. ⌋

$\{R|\vec{\tau}\} \in$ Euclidean group.

$$\{R|\vec{\tau}\}\cdot\vec{r} = R\cdot\vec{r} + \vec{\tau}$$

$$\{R_1|\vec{\tau}_1\}\{R_2|\vec{\tau}_2\}\vec{r} = \{R_1|\vec{\tau}_1\}(R_2\vec{r} + \vec{\tau}_2)$$
$$= R_1R_2\vec{r} + (R_1\vec{\tau}_2 + \vec{\tau}_1)$$
$$= \{R_1R_2 | R_1\vec{\tau}_2 + \vec{\tau}_1\}\vec{r}$$

matrix rep. $\begin{pmatrix} \boxed{R} & \tau \\ & 1 \end{pmatrix}$ The 3 dims for the rotational part is invariant.

When is a rep (T, V) completely reducible?

Proposition: Let (T, V) be a unitary rep. of on an inner product space V, and $W \subset V$ is an invariant subspace. The $W^{\perp}$ is an invariant subspace.

$$(W^{\perp} = \{ y \in V \mid \langle y, x\rangle = 0 \ \forall x \in W \})$$

$\forall g \in G$, $y \in W^{\perp} \overset{?}{\Rightarrow} T(g)y \in W^{\perp}$

$\langle T(g)y, x\rangle = \langle y, T(g)^{+}x\rangle = \langle y, T(g^{-1})x\rangle \xlongequal{T(g)x \in W} 0$

$\Rightarrow$ $T(g)y \in W^{\perp}$, $\forall g \in G$

$\Rightarrow$ $W^{\perp}$ invariant subspace.

Corollaries:

1. Finite-dimensional unitary reps are always completely reducible.

$$V \text{ reducible? } \xrightarrow{Y} V \cong W \oplus W^{\perp} \xrightarrow{W^{\perp}?} W^{\perp} \cong W' \oplus W'^{\perp}$$

$$V \cong W \oplus W' \oplus W'^{\perp}$$

$$\rightarrow V \cong W \oplus W' \oplus W'' \oplus \cdots$$

Why finite dimensional? Consider irreps of $\mathbb{R}$

$\rho_k(a) = e^{ika}$. $k \in \mathbb{R}$.

2. For compact groups, reps are unitarizable

$\Rightarrow$ completely reducible.

3. Finite G. $L^2(G)$ is completely reducible

⌈ Recall that e.g. the Left rep on $L^2(G)$

$$L_g \cdot \delta_h = \delta_{gh}. \qquad \delta\text{ - basis.}$$

$|G|$-dimensional rep. ⌋

Example of reg. rep. of S_3

$$SMS^{-1} = \begin{pmatrix} O & & \\ & O & \\ & & O \end{pmatrix}$$

$$\begin{cases} \chi(e) = |S_3| = 6 \\ \chi(g \neq e) = 0 \end{cases}$$

We will not try to find the S explicitly here.
but focus on the dimensions of the irreps.

χ_V	$1\,(\,)$ [I]	$3\,(12)$ [II]	$2\,(123)$ [III]
V_1	1	1	1
V_1'	1	-1	1
V_2	2	0	-1

$|S_3| = 6$

① recall the canonical rep on $\mathbb{R}^3 = \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

a. trivial irrep: $V_1 \cong \{\sum e_i\}$

$P_1(\phi)\,\vec{v} = \vec{v}$

b. its orthogonal completement:

$V_2 = V_1^{\perp} = \text{span}.\{e_1 - e_2,\ e_2 - e_3\}$

② recall the "sgn" hom:

$$\begin{aligned} \text{sgn} \quad S_3 &\longrightarrow \mathbb{Z}_2 \\ \phi &\longmapsto \text{sgn}\,\phi \end{aligned}$$

$\Rightarrow\ L^2(S_3) \cong V_1 \oplus V_1' \oplus V_2$? × $\cong V_1 \oplus V_1' \oplus 2V_2$

Some sanity check:

$\dim(V_1 \oplus V_1' \oplus V_2) = 1 + 1 + 2 \neq 6$. we are missing something

Later we will know that these are the only 3 irreps of S_3 ($\because$ # irreps = # conj. classes)

How to find the missing pieces?

use character: $\chi_{reg}(e) = 6$

$\chi_{reg}(g \neq e) = 0$

$$V^{reg} \cong x V_1 \oplus y V_1' \oplus z V_2$$

$$\Rightarrow \chi_{reg} = x \cdot \chi_1 + y \cdot \chi_{1'} + z \chi_2$$

$$\left.\begin{aligned} e: &\quad x + y + 2z = 6 \\ [(12)]: &\quad x - y = 0 \\ [(123)]: &\quad x + y - z = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 1 \\ y &= 1 \\ z &= 2 \end{aligned}$$

$$\Rightarrow \quad V^{reg} \cong V_1 \oplus V_1' \oplus 2V_2$$

This is a form of <u>isotypic decomposition</u>.

Assume that the set of irreps (up to isomorphism) of G is countable, choose a representative $(T^{(\mu)}, V^{(\mu)})$ for each isomorphism class

$$V \cong \oplus_\mu \oplus_{i=1}^{a_\mu} V^{(\mu)}$$

a_μ is the number of times $V^{(\mu)}$ appears in the decomposition. $\oplus_{i=1}^{a_\mu} V^{(\mu)}$ is the <u>isotypical component</u> of V belonging to μ. also, note that we can identify for $a_\mu \neq 0$

$$\underbrace{V^{(\mu)} \oplus V^{(\mu)} \oplus \cdots \oplus V^{(\mu)}}_{a_\mu} \cong \underline{\underline{K^{a_\mu} \otimes V^{(\mu)}}} =: a_\mu V^{(\mu)}$$

Explicit isomorphism:

$V^{(\mu)} = \mathrm{span}\{v_i,\ i = 1, \cdots n_\mu\}$

$K^{a_\mu} = \mathrm{span}\{e_1, \cdots e_{a_\mu}\}$ standard basis on K^{a_μ}.

$\bigoplus^{a_\mu} V^{(\mu)} = \mathrm{span}\{(v^{(1)}, v^{(2)}, \cdots v^{(a_\mu)}),\ v^{(i)} \in V^{(\mu)}\}$, $\dim = n_\mu^{a_\mu}$

choose basis:

$$e_{i,\mu} = (0, 0, \cdots \underset{i\text{th position}}{\underline{v_\alpha}}, 0, 0)$$

$$\phi:\ K^{a_\mu} \otimes V^{(\mu)} \longrightarrow \bigoplus_{i=1}^{a_\mu} V^{(\mu)}$$

$$e_i \otimes v_\alpha \longmapsto (0, 0, \cdots \underset{i\text{th}}{\underline{v_\alpha}}, 0, 0) \equiv e_{i\alpha}$$

it is easy to see it is an isomorphism (the map is invertible)

Correspondingly, on $K^{a_\mu} \otimes V^{(\mu)}$, define group action

$$g \in G:\quad e_i \otimes v \longmapsto e_i \otimes T^{(\mu)}(g)\, v.$$

i.e. the rep on $K^{a_\mu} \otimes V^{(\mu)}$ can be defined as

$$T(g) = \mathbb{1}_{K^{a_\mu}} \otimes T^{(\mu)}(g)$$

it acts trivialy on the "multiplicity/degeneracy space" K^{a_μ}

in matrix form:

$$T(g):\ \psi_{\mu,i,\alpha} \longmapsto \sum_{j=1}^{n_\mu} M^{\mu}(g)_{ji}\, \psi_{\mu,j,\alpha}$$

irrep label

$i = 1, \dots n_\mu$

$\alpha = 1, \dots a_\mu$

degeneracy index.