Recap.

1. Haar measure $\Rightarrow$ compact groups. define

$$\langle u, v \rangle_2 = \int_G d\mu(g) \langle \pi(g) u, \pi(g) v \rangle$$

$\Rightarrow$ unitarizable.

2. Regular rep: $L^2(G) := \{ f : G \to \mathbb{C} \mid \int_G d\mu(g) |f(g)|^2 < \infty \}$

defined via group action $G \times G$ on $\mathcal{F} = \mathrm{Map}(G, \mathbb{C})$.

$$(g_1, g_2) \cdot f(g) = f(g_1^{-1} g g_2) \ (= f'(g))$$

(Corresponding to $G \times G$ action on G:

$$(g_1, g_2) \cdot g = g_1 g g_2^{-1}$$

Why rep?

① identity: $(e, e) f(g) = f(g) \quad \forall f \in \mathcal{F}$

② grp. mult. $\{ (g_1', g_2') [(g_1, g_2) f] \} (g)$

$$= [(g_1, g_2) f] (g_1'^{-1} g g_2')$$

$$= f(g_1^{-1} g_1'^{-1} g g_2' g_2)$$

$$= (g_1' g_1, g_2' g_2) \cdot f(g)$$

$L^2(G)$ actually rep of $G \times G$.

restrict to $G \times \{1\}$ or $\{1\} \times G$ to get

left or right reg. rep. of G.

2. For a rep (T, V). $\mathrm{End}(V)$ is also a rep of $G \times G$: $S \in \mathrm{End}(V)$, then define action similar to above:

$$(g_1, g_2) \cdot S := T(g_1) \cdot S \cdot T(g_2)^{-1}$$

also a rep. relation? For finite-dim V:

$$i: \mathrm{End}(V) \longrightarrow L^2(G)$$
$$S \longmapsto f_S = \mathrm{Tr}_V(S\,T(g))$$

intertwiner / $G \times G$ equivariant:

$$(h_1, h_2) \cdot f_S(g) = f_{(h_1, h_2) \cdot S}(g)$$

8.6. Regular representation (cont.)

Equip V with an ordered basis $\{v_i\}$

$$T(g)\cdot v_i = \sum_j M(g)_{ji} v_j$$

and take S to be the matrix unit e_{ij}

($[e_{ij}]_{ab} = \delta_{ia}\delta_{jb}$, a basis of $\mathrm{End}(V)$)

$$f_S = \mathrm{Tr}_V (S\, T(g^{-1}))$$

$$= \mathrm{Tr} (\sum_b \delta_{ia}\delta_{jb} M_{bc}(g^{-1}))$$

$$= \sum_{ac} [\delta_{ia} M_{jc}(g^{-1})] \delta_{ac}$$

$$= M_{ji}(g^{-1})$$

($f_S = M_{ij}(g)$ if replace V by its dual space V^*.

recall $M^*(g) = [M(g^{-1})]^{tr} = M(g)^{tr,-1}$)

$\Rightarrow$ f_S's are linear combinations of matrix elements of rep. of G.

What's the point of all these?

1. Matrix elements of <u>any representation</u> (T, V) appear as L^2-functions on the group.

2. $L^2(G)$ contains <u>all reps</u> of G.

   We know that for finite groups. $L^2(G)$ is finite dimensional. So there are finitely many "essentially different" reps. or any rep is built out of a finite set of "basic building blocks".

8.7 Reducible & irreducible representations

Recall the direct sum of reps.

$$T_{V\oplus W} = T_V \oplus T_W$$

$$M_{V\oplus W} = \left(\begin{array}{c|c} M_V & 0 \\ \hline 0 & M_W \end{array}\right)$$

We can imagine that "large" reps can be "reduced" to smaller building blocks

Definition Let $W \subset V$ be a linear subspace of carrier space V of a group rep. $T: G \to GL(V)$. Then W is invariant under T. a.k.a an invariant subspace if $\forall g \in G$. $w \in W$.

$$T(g)\, w \in W.$$

Example

1. $\{\vec{0}\}$ & V

2. $\mathbb{R}^3$ under $SO(2)_z$: xy plane is a subspace

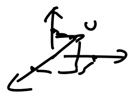
Quiz here: (other planes at finite z_0 are not)

3. canonical rep. of S_n:

$$T(\phi): \vec{e}_i \to \vec{e}_{\phi(i)}$$

Then $\vec{v} = \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_n$ is invariant

$$T(\phi)\vec{v} = T(\phi)\sum_i \vec{e}_i = \sum_i \vec{e}_{\phi(i)} = \vec{v}$$

in $\mathbb{R}^3$: 

diagonal vector

4. Mat rep.

$$M: G \to GL(n, K)$$

M_{ij} as a function: $G \to K$

$$g \mapsto M_{ij}(g)$$

The linear span of M_{ij} with <u>fixed i</u>

$$R_i := \text{span}\{M_{ij},\ j = 1, \cdots n\}$$

<u>right action</u>:

$$\underbrace{(R(g)\cdot M_{ij})}_{\substack{M' \\ \text{a function}}}(h) = M_{ij}(hg)$$

$$= \underbrace{\sum_s M_{sj}(g)}_{\text{coefficients}} M_{is}(h)$$

$\Rightarrow R_i$ is an invariant subspace

<u>left action</u>:

$$L_j := \text{span}\{M_{ij},\ i = 1, \cdots n\}$$

is also invariant

$\Rightarrow$ $\mathcal{LR} = \mathrm{span}\{ \mu_{ij} ,\quad i,j = 1, \dots n \}$ subspace of $L^2(G)$

is invariant under $G \times G$ - action

$$((g_1, g_2) \cdot f)(h) = f(g_1^{-1} h g_2)$$

note under left G action,

$$\mathcal{LR} \cong \oplus_i^n \mathcal{L}_i$$

Remarks

1 (T, V) a rep. $\exists\, W \subset V$ an invariant subspace. then we can restrict T to W.

$(T|_W, W)$ is a <u>subrepresentation</u> of (T, V)

$$T|_W(g) = T(g)|_W$$

We will write T instead of $T|_W$.

2. if T is unitary on V then it is unitary on W.

$$\langle T v_1, T v_2 \rangle = \langle v_1, v_2 \rangle \quad \forall\, v_i \in V.$$

Definition. A representation (T, V) is **reducible**

if there is a proper, nontrivial invariant subspace $W \subset V$ ($W \neq 0, V$)

If V is not reducible, it is an **irreducible representation** ("irrep")

Remarks.

1. $\forall v \in V$. $\text{span}\{T(g)v, \forall g \in G\}$ is an invariant subspace.

   If T is an irrep. it is V.

   such a vector is called a cyclic vector.

Note: the existence does not imply that the representation is irreducible

Consider e_1 in the **permutation representation**.

$\mathbb{I}e_i$ is a proper, nontrivial invariant subspace

2. (T, W) a subrep of (T, V)

Choose an ordered basis

$\{w_1, \cdots w_k\}$

Then it can be completed to an ordered basis of V

$\{w_1, \cdots w_k, u_{k+1}, \cdots u_n\}$

$$T(g)(w_i) = (M_{11}(g))_{ji}\, w_j + (M_{21}(g))_{ai}\, u_a$$

$$T(g)(u_a) = (M_{12}(g))_{ja}\, w_j + (M_{22}(g))_{ba}\, u_b$$

i.e. $(w, u)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$

W invariant $\Rightarrow M_{21} = 0$

$$\Rightarrow T(g)(w_i) = \sum_j M_{11}(g)_{ji}\, w_j$$

$$\begin{pmatrix} M_{11}^{g_1} & M_{12}^{g_1} \\ 0 & M_{22}^{g_1} \end{pmatrix}\begin{pmatrix} M_{11}^{g_2} & M_{12}^{g_2} \\ 0 & M_{22}^{g_2} \end{pmatrix} = \begin{pmatrix} \boxed{M_{11}^{g_1} M_{11}^{g_2}} & M_{11}^{g_1} M_{12}^{g_2} + M_{12}^{g_1} M_{22}^{g_2} \\ 0 & M_{22}^{g_1} M_{22}^{g_2} \end{pmatrix}$$

M_{11} is a rep on W

M_{22} is not a rep on $V \backslash W$.

What if we want to further simplify it ?

If we define a change of basis $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -S \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(w, u)\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = (w, wS + u) \equiv (w, u')$$

$$\begin{pmatrix} 1 & -S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} M_{11}(g) & M_{12}(g) \\ 0 & M_{22}(g) \end{pmatrix} = \begin{pmatrix} M_{11}(g) & M_{12}(g) - S M_{22}(g) \\ 0 & M_{22}(g) \end{pmatrix}$$

we require $M_{12}(g) - S M_{22}(g) = 0 \quad \forall g \in G$.

This puts a stronger restriction on the structure of the representation.

3. quotient space. V/W.

$$v_1 \sim v_2 \text{ iff } v_1 - v_2 \in W.$$

$$T(g)(v + W) = T(g)(v) + W$$

$$\begin{aligned} \Rightarrow T(g_1) T(g_2)(v+W) &= T(g_1)(T(g_2) v + W) \\ &= T(g_1) T(g_2) v + W \\ &= [T(g_1) T(g_2)](v + W) \end{aligned}$$

u_a define a basis for V/W as $u_a + W$. The rep looks like M_{22} wrt this basis.

<u>Definition</u> A representation T is called completely reducible if it is isomorphic to a direct sum of representations.

$$W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

where W_i are irreps. Thus there is a basis in which the matrices look like

$$M(g) = \begin{pmatrix} M_{11}(g) & 0 & 0 & \cdots \\ 0 & M_{22}(g) & \cdots & \\ 0 & \ddots & M_{33}(g) & \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

irreps are completely reducible.

reducible but not completely $\Rightarrow$ "indecomposable"