Recap:

① unitary & unitarizable

inner product: $\langle \cdot , \cdot \rangle$

(T, V): $\langle T(g)u, T(g)v \rangle = \langle u, v \rangle \quad \forall g \in G.\ u, v \in V$

is a unitary rep

if $T(g) = A\, U(g)\, A^{-1} \quad \forall g \in G.\ U$ unitary

$\Rightarrow$ T unitarizable.

② finite group:

A. work on the operators.

$$H = \sum_g T(g)^+ T(g)$$

$\tilde{T}(g) = H^{\frac{1}{2}}\, T(g)\, H^{-\frac{1}{2}}$, then

$\tilde{T}(g)^+ \tilde{T}(g) = \mathbb{1}$.

B. redefine $\langle \cdot , \cdot \rangle_2 = \langle \cdot \ . H \cdot \rangle / |G|$

③ extend ideas from finite groups: $\frac{1}{|G|} \sum_g \to \int_G d\mu(g)$

Haar measure: $\int_G d\mu(g)\, f(g) = \int_G d\mu(g)\, f(hg) \quad \forall h \in G.$

(left)

$d\mu(g) = d\mu(h^{-1}g)$

$$\begin{cases} (\mathbb{R}, +) \cdot d\mu_{\mathbb{R}}(x) = c \cdot dx \\ (\mathbb{R}^+_{>0}, \times)\ d\mu_{\mathbb{R}^*}(x) = c \cdot \frac{dx}{x} \\ GL(n, \mathbb{R})\ d\mu(g) = |\det g|^{-n} \prod dg_{ij} = \prod dg_{ij} \text{ for } \underline{SL} \end{cases}$$

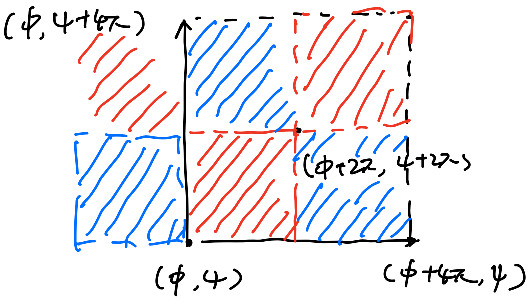
$SU(2)$: $d\mu_{SU(2)} = \frac{1}{16\pi^2} d\phi \sin\theta d\theta d\psi$

if we define $g(\phi, \theta, \psi) = e^{i\frac{\sigma_z}{2}\phi} e^{i\frac{\sigma_x}{2}\theta} e^{i\frac{\sigma_z}{2}\psi}$

i.e. $\alpha = e^{\frac{i}{2}(\phi+\psi)} \cos\frac{\theta}{2}$ $\qquad \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$

$\beta = i e^{\frac{i}{2}(\phi-\psi)} \sin\frac{\theta}{2}$

fix θ : $(\phi, \psi) \sim (\phi+4\pi, \psi) \sim (\phi, \psi+4\pi) \sim (\phi+2\pi, \psi+2\pi)$



There is some freedom in choosing the integration domains

we take $\theta \in [0, \pi)$ $\phi \in [0, 2\pi)$ then $\psi \in [0, 4\pi)$

Q if we know $d\mu_G$, do we get $d\mu_{G/H}$ "for free"?

$\pi : G \to G/H \qquad (H \subset G)$

$g \mapsto gH$

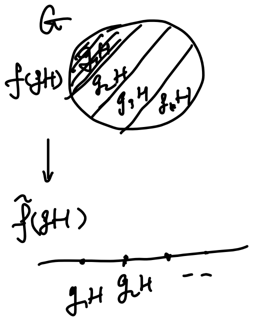
$SU(2) \longrightarrow SO(3) \cong SU(2)/\mathbb{Z}_2$

$u \longmapsto u\{\pm 1\} = \{u, -u\}$

For a function on G/H, we can "lift" it to be a function on G that is invariant under H action. for any $\tilde{f}: G/H \to \mathbb{C}$, one can define

$f: G \to \mathbb{C}$. s.t. $f(gh) = \tilde{f}(\pi(g)) \ (\forall h \in H)$

conversely, we view $\tilde{f}$ as a projection of f onto G/H



Now think of $\int_G f(g)\, d\mu_G(g)$. this can be formally written as

$$\int_G f(g)\, d\mu_G(g) = \int_{G/H} \underbrace{\left[\int_H f(gh)\, d\mu_H(h)\right]}_{\pi_H f(g)} d\mu_{G/H}(gH)$$

we use this to define $d\mu_{G/H}$.

Then $d\mu_{G/H}$ is just the projection of $d\mu_G$ onto G/H.

in $SU(2) \longrightarrow SO(3) \cong SU(2)/\mathbb{Z}_2$

$(\phi, \theta, \psi) \longmapsto \{(\phi, \theta, \psi), (\phi, \theta, \psi + 2\pi)\}$ choose one representative in $\underline{gH}$

so only needs to restrict ψ to $[0, 2\pi)$

in $SU(2) \longrightarrow S^2 \cong SU(2)/U(1)$

think of $g_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $SU(2)$ action by conjugation. then its stabilizer

$$d g_0 d^\dagger = g_0 \Rightarrow \left\{ g = \begin{pmatrix} e^{i\gamma} & \\ & e^{-i\gamma} \end{pmatrix}, \gamma \in [0, 2\pi) \right\} \cong U(1)$$

its orbit $\cong S^2$ $\quad \{ g = e^{i \frac{\sigma_z}{2} \psi}, \ \psi \in [0, 4\pi) \}$

then $d\mu_{G/H} = \frac{1}{4\pi} d\phi \sin\theta\, d\theta$ by integrating out ψ

8.5 . Haar measure (cont..)

Examples 8: left Haar measure $\neq$ right for LC groups

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R},\ x > 0 \right\}$$

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} \\ 0 & 1 \end{pmatrix} \in G$$

or a non abelian multiplication: $(u, v)\cdot(x, y) = (ux, uy+v)$
$(x, y)(u, v) = (ux, vx+y)$

① left invariance: $g \mapsto g_0 g$

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ux & uy+v \\ 0 & 1 \end{pmatrix}$$

$$dx\,dy \mapsto u^2 dx\,dy$$

$$d\mu(x, y) = x^{-2} dx\,dy$$

check: $d\mu((u, v)(x, y)) = d\mu\left(\frac{d(ux)\,d(uy+v)}{(ux)^2} \right) = d\mu(x, y)$

② right invariance: $(x, y)(u, v) = (ux, vx+y)$

$$dx\,dy \mapsto u\,dx\,dy$$

$$d\mu(x, y) = x^{-1} dx\,dy$$

$$d\mu[(x, y)(u, v)] = \frac{u\,dx\,dy}{(ux)} = d\mu(x, y)$$

Now that we have Haar measures for <u>compact groups</u>, then we can unitarize them! (recall why we start talking about Haar measures).

Proposition: If (T, V) is rep of a compact group G, and V is an inner product space

$\Rightarrow$ (T, V) is unitarizable.

We do exactly as we did for finite groups earlier:

If T is not already unitary w.r.t inner product $\langle \cdot \rangle_1$, then we can define a new inner product

$$\langle v, w \rangle_2 := \int_G \langle T(g) v, T(g) w \rangle_1 \, d\mu(g)$$

Then $\underline{\langle T(g) v, T(g) w \rangle_2 = \langle v, w \rangle_2}$

$$\left(\begin{aligned} \langle T(h) v, T(h) w \rangle_2 &= \int_G \langle T(hg) v, T(hg) w \rangle_1 \, d\mu(g) \\ &\overset{\text{Haar}}{=} \int_G \langle T(g) v, T(g) w \rangle_1 \, d\mu(g) \end{aligned} \right)$$

$$= \langle v, w \rangle_2$$

Remarks:

1 det: $GL(n, K) \longrightarrow K^*$ non compact.

$$A \longmapsto \det A$$

$$\langle \det A \, z_1, \det A \, z_2 \rangle = |\det A|^2 \, \bar{z}_1 z_2$$

not unitarizable. (infinite volume)

2. Corollary for matrix representations.

Compact groups. $\exists A$ s.t. the matrix rep

$$u(g) = A \, M(g) A^{-1} \quad \forall g$$

where $u(g)$ unitary

Define usual inner product on $\mathbb{C}$

two set of basis $\{e_i^1\}$, $\{e_i^2\}$, $\{e^2\}$ is ON.

$$e_i^{(1)} = \sum_k A_{ki} e_k^{(2)}$$

$$\langle e_i^{(1)}, e_j^{(1)} \rangle = \sum_{kk'} \langle A_{ki} e_k^{(2)}, A_{k'j} e_{k'}^{(2)} \rangle$$

$$= \sum_{kk'} \overline{A_{ki}} A_{k'j} \delta_{kk'}$$

$$= \sum_k \overline{A}_{ki} A_{kj}$$

$$= (A^+ A)_{ij}$$

if u is unitary w.r.t. $\{e_i^{(2)}\}$

then the unitary rep in $\{e_i^{(1)}\}$

is $\quad \hat{u} = A^{-1} u A$

$$\hat{u}\, e_i^{(1)} = \sum_j \hat{u}_{ji} e_j^{(1)} = \sum_{jk} \hat{u}_{ji} A_{kj} e_k^{(2)}$$

$$= \sum_k (A\hat{u})_{ki} e_k^{(2)}$$

$$= \sum_k (uA)_{ki} e_k^{(2)}$$

$$\langle \hat{u} e_i^{(1)}, \hat{u} e_j^{(1)} \rangle = \sum_{kk'} \overline{(uA)_{ki}} (uA)_{k'j} \delta_{kk'}$$

$$= \sum_k [(uA)^+]_{ik} (uA)_{kj}$$

$$= (A^+ u^+ u A)_{ij}$$

$$= (A^+ A)_{ij} = \langle e_i^{(1)}, e_j^{(1)} \rangle$$

(quite often we work on ON basis in physics. the choice of unitarization approach depends on basis.

ON: change operators

not ON: basis, or both.

8.6 The Regular representation.

Let G be a group. Then there is a left action of $G \times G$ on G:

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1}):$$

$$(g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1}$$

and hence an induced action on $Map(G, \mathbb{C})$

$$((g_1, g_2) \cdot f)(h) := f(g_1^{-1} h g_2)$$

which converts the vector space of functions $f: G \to \mathbb{C}$ into a representation space for $G \times G$.

Recall for induced g action:

$$\tilde{\Phi}(g, F)(x) = F(\phi(g^{-1}, x))$$

$$\tilde{\Phi}(g_1, \tilde{\Phi}(g_2, F))(x) = \tilde{\Phi}(g_2, F)(\phi(g_1^{-1}, x)) = F(\phi(g_2^{-1}, \phi(g_1^{-1}, x))$$
$$= F(g_2^{-1} g_1^{-1}, x)$$
$$= F(\phi((g_1 g_2)^{-1}, x))$$
$$= \tilde{\Phi}(g_1 g_2, F)(x)$$

$$\{[(g_1, g_2)(g_3, g_4)]f\}(h) = [(g_1 g_3, g_2 g_4) f](h)$$
$$= f(g_3^{-1} g_1^{-1} h g_2 g_4)$$

$$\{(g_1, g_2) \cdot [(g_3, g_4) f]\}(h) = [(g_3, g_4) f](g_1^{-1} h g_2)$$
$$= f(g_3^{-1} g_1^{-1} h g_2 g_4)$$

This can be viewed as a group homomorphism

$$G \times G \longrightarrow \underline{\mathrm{End}}(\{\varphi\psi\}) := \mathrm{Hom}(\{\varphi\}, \{\varphi\})$$ cf. Aut( )

vector space $\varphi: G \to \mathbb{C}$ becomes a representation space for $G \times G$.

Now, equip G with a left and right-invariant Haar measure, and consider completion of all square integrable functions on G:

$$L^2(G) = \{ f: G \to \mathbb{C} \mid \underbrace{\int_G |f(g)|^2 \, d\mu(g)}_{\langle f, f\rangle} < \infty \}$$

i.e. the Hilbert space.

Then $G \times G$ action preserves the L^2-property because of the left & right Haar measure

<u>Definition</u> The representation $L^2(G)$ is known as the regular representation of G.

If we restrict $G \times G$ to subgroups $G \times \{1\}$ or $\{1\} \times G$, then $L^2(G)$ becomes a representation of G:

$$(L(h) \cdot f)(g) := f(h^{-1} g)$$

then it is the <u>left regular representation</u>

$$(R(h) \cdot f)(g) = f(gh)$$

defines the <u>right regular representation</u>

Note: $L(h)$, $R(h)$ acts on the function space on the left.

Example 1. $G = \mu_3 = \{1, \omega, \omega^3\}$ $\omega = e^{i\frac{2}{3}\pi}$

assign a basis of $L^2(G)$: $\delta_j(\omega^k) = \begin{cases} 1 & j = k \bmod 3 \\ 0 & \text{else} \end{cases}$

$$(L(\omega)\cdot\delta_0)(g) = \delta_0(\omega^{-1}g) = \delta_1(g)$$

$$L(\omega)\delta_0 = \delta_1$$
$$L(\omega)\delta_1 = \delta_2$$
$$L(\omega)\delta_2 = \delta_0$$

$$L(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This is the "reg. rep" we talked before for finite groups:

$$g\delta_{g_0} = \delta_{gg_0}$$

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$$L(1)g = g \qquad L(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L(\omega)1 = \omega$$
$$L(\omega)\omega = \omega^2$$
$$L(\omega)\omega^2 = 1$$

$$L(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Suppose (T, V) is a finite-dim. representation of G.

We can define $G \times G$ action on $\mathrm{End}(V) := \mathrm{Hom}(V, V)$

$\forall S \in \mathrm{End}(V)$:

$$(g_1, g_2) \cdot S := T(g_1) \cdot S \cdot T(g_2)^{-1}$$

How are the two representation space related?

For finite-dimensional V, we can define a map

$$\iota : \mathrm{End}(V) \longrightarrow L^2(G)$$

$$S \longmapsto f_S$$

$$f_S := \mathrm{Tr}_V(S\,T(g^{-1}))$$

which is $G \times G$ equivariant. (ι is an intertwiner)

$$\begin{tikzcd} \mathrm{End}(V) \arrow[r, "\iota"] \arrow[d, "T_{\mathrm{End}(V)}"] & \mathrm{Map}(G, \mathbb{C}) \arrow[d, "T_{\mathrm{reg.\,rep}}"] \\ \mathrm{End}(V) \arrow[r, "\iota"] & \mathrm{Map}(G, \mathbb{C}) \end{tikzcd}$$

$\because$: $(h_1, h_2) f_S(g) = f_S(h_1^{-1} g h_2)$

$$= \mathrm{Tr}_V(S\,T(h_2^{-1} g^{-1} h_1))$$

$$= \mathrm{Tr}_V(S\,T(h_2)^{-1}\,T(g^{-1})\,T(h_1))$$

$$= \mathrm{Tr}_V(\underbrace{T(h_1)\,S\,T(h_2)^{-1}}\,T(g^{-1}))$$

$$= \mathrm{Tr}_V((h_1, h_2) \cdot S\,T(g^{-1}))$$

$$= f_{(h_1, h_2)\cdot S}(g) \quad = \Lsh$$

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ (h_1, h_2)\cdot S \arrow[d] & & f_S \arrow[d] \\ f_{(h_1,h_2)\cdot S} & = & (h_1, h_2) f_S \end{tikzcd}$$