Recap: representation theory

1. $G \xrightarrow{\text{rep.}} GL(V) \xrightarrow[\cong]{\text{ordered basis } \{\hat{v}_i\}} GL(n, k)$

$g \longmapsto T(g)$ $\qquad T(g)\hat{v}_i = \sum_j M(g)_{ji} \hat{v}_j$

$V = k^n$ carrier space / rep. space

n: dim of rep.

2. intertwiner: equivariant linear map $V_1 \longrightarrow V_2$

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

$A T_1(g) = T_2(g) A$

$A \in \mathrm{Hom}_G(V_1, V_2)$

if A invertible. $T_1(g) A^{-1} = A^{-1} T_2(g)$

$$\begin{tikzcd} V_1 \arrow[d, "T_1"'] & V_2 \arrow[l, "A^{-1}"'] \arrow[d, "T_2"] \\ V_1 & V_2 \arrow[l, "A^{-1}"] \end{tikzcd}$$

3. equivalent rep. $\quad T_2(g) = A T_1(g) A^{-1}$

4. character as a class function $\quad \chi_T(g) = \mathrm{Tr}_V(T(g))$

$\chi_T(hgh^{-1}) = \chi_T(g) \qquad g, h \in G$

5. $T_1 \oplus T_2 \quad \chi_\oplus = \chi_1 + \chi_2 \qquad T_1 \otimes T_2 \quad \chi_\otimes = \chi_1 \cdot \chi_2$

§.4 Unitary representations

Let V be a complex vector space over $\mathbb{C}$.

Define the inner product on V as a sesquilinear map $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$. obeying

(1) $\langle v, * \rangle$ is linear for all fixed v.

(2) $\langle w, v \rangle = \overline{\langle v, w \rangle}$

(3) $\langle v, v \rangle \geq 0$. $= 0$ iff $v = 0$

sesquilinear: linear in one argument. conj. linear in another

$$\left(\begin{aligned} &\langle v, \alpha_1 w + \alpha_2 w_2 \rangle = \alpha_1 \langle v_1, w_1 \rangle + \alpha_2 \langle v_1, w_2 \rangle \\ &\langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \overline{\alpha_1} \langle v_1, w \rangle + \overline{\alpha_2} \langle v_2, w \rangle \end{aligned} \right)$$

Definition. Let V be an inner product space

A unitary rep is a rep (V, U) s.t. $\forall g \in G$ $U(g)$ is a unitary operator on V. i.e.

$$\langle U(g) v, U(g) w \rangle = \langle v, w \rangle \quad \forall v, w \in V \quad \forall g \in G.$$

Definition. If a rep (V, T) is equivalent to a unitary rep. then it is said to be unitarizable.

How to unitarize? $\langle \rho(g)u, \rho(g)v \rangle = \langle u, v \rangle$

$\Rightarrow \langle u, \rho^+(g)\rho(g)v \rangle = \langle u, v \rangle \Rightarrow \rho^+(g)\rho(g) = \mathbb{1}$

average over the group: $\frac{1}{|G|}\sum_g \langle \rho(g)u, \rho(g)v \rangle := \langle u, v \rangle_G$

$$\langle \rho(h)u, \rho(h)v \rangle_G = \frac{1}{|G|}\sum_g \langle \underbrace{\rho(g)\rho(h)}_{\rho(gh)}u, \underbrace{\rho(g)\rho(h)}_{\rho(gh)}v \rangle = \langle u, v \rangle_G$$

Equivalently $\langle u, v \rangle_G := \langle u, Hv \rangle / |G|$. where

$$H = \sum_{g \in G} \rho^+(g)\rho(g)$$, which statisfies

Hermitian adjoint

$$\rho(g)^+ H \rho(g) = \sum_h \rho^+(g)\rho^+(h)\rho(h)\rho(g) = \sum_h \rho(hg)^+\rho(hg) = H$$

H is positive-definite and Hermitian. the $\exists\, U$. st.

$$UHU^+ = \Lambda = \mathrm{diag}(\lambda_1 \cdots \lambda_n) \quad (\forall \lambda_i > 0)$$

Define the square root $H^{\frac{1}{2}} = U\Lambda^{\frac{1}{2}}U^+$. then

If we "normalize" $\rho \longrightarrow H^{\frac{1}{2}}\rho H^{-\frac{1}{2}} := \tilde{\rho}$. then

$$\tilde{\rho}^+(g)\tilde{\rho}(g) = (H^{-\frac{1}{2}}\rho^+(g)H^{\frac{1}{2}})(H^{\frac{1}{2}}\rho(g)H^{-\frac{1}{2}})$$

$$= H^{-\frac{1}{2}}\underbrace{(\rho^+ H \rho)}_{=H}H^{-\frac{1}{2}}$$

$$= \mathbb{1}$$

this works for all finite groups: $\frac{1}{|G|}\sum_g$ is well defined.

① Reps of finite groups are always unitarizable.

② How to define "$\frac{1}{|G|}\sum_g$" for infinite / continuous groups? $\frac{1}{|G|}\sum_{g \in G} \longrightarrow \int_G dg$?

8.5 . Haar measure (a.k.a invariant integration)

Consider a function $f : G \to \mathbb{C}$. $f \in \mathrm{Map}(G, \mathbb{C})$

$$\langle f \rangle := \frac{1}{|G|} \sum_{g \in G} f(g) \quad \text{“}\Longrightarrow\text{”} \quad \underline{\int_G} dg\, f(g)$$

$$\int_G dg \in \left(\mathrm{Map}(G, \mathbb{C})\right)^{\vee} = \mathrm{Hom}\left(\mathrm{Map}(G, \mathbb{C}), \mathbb{C}\right)$$

$$\{f\}$$

$$\int_G dg : f \mapsto \langle f \rangle$$

For finite group. $\underline{\frac{1}{|G|} \sum_{g \in G} f(hg) = \frac{1}{|G|} \sum_{g \in G} f(g)}$

invariant under left translation $L_h : g \mapsto hg$

We require similarly for $\int_G dg$:

$$\underline{\int_G f(hg)\, dg = \int_G f(g)\, dg} \quad (\forall h \in G)$$

left invariance condition.

Left Haar measure.

(right Haar measure : $\int_G f(gh)\, dg = \int_G f(g)\, dg$)

1. For a finite group, left and right invariant measures are unique up to an overall scale.

$$\frac{1}{|G|} \sum_g f(hg) = \frac{1}{|G|} \sum_g f(gh)$$

holds also for compact Lie groups.

in general physics context: subset of $\mathbb{C}^m$.

compact $\Longleftrightarrow$ $\underline{\text{closed \& bounded}}$

$$U(n) = \{ A \in GL(n, \mathbb{C}) \mid \underline{A^\dagger A = \mathbb{1}} \} \subset \mathbb{C}^{n^2}$$

$$\sum_j (A^\dagger)_{ij} A_{ji} = 1$$

$$\Rightarrow \sum_j |A_{ji}|^2 = 1 \Rightarrow |A_{ji}| \leq 1 \quad \forall i, j$$

other examples: $Sp(n) \cong U(2n) \cap Sp(2n, \mathbb{C})$

$Sp(1) \cong SU(2)$

<u>non-compact.</u> $O(1, d)$

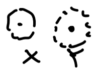
$Sp(2n, K) \to \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \quad B^T = B$

$GL(n, K)$

2. <u>Locally compact & Hausdorff</u>

LC: each point's neighborhood looks compact.

$\mathbb{R}^n$ not compact, but LC

Hausdorff: points separable  $X \cap Y = \phi$

There exists a left invariant measure on G, which is unique up to scale

(similar for right-invariance)

But <u>Left ≠ right.</u> (see later examples)

Examples

1. $G = (\mathbb{R}, +)$

$$\int_G dg\, f(g) = \int_G dg\, f(g+a) \qquad (a \in \mathbb{R})$$

$$\Rightarrow c\int_{-\infty}^{+\infty} dx\, f(x)$$

$$c\int_{-\infty}^{\infty} dx\, f(x+a) = c\int_{-\infty}^{\infty} d(x+a)\, f(x+a) = c\int_{-\infty}^{\infty} dx\, f(x)$$

2. $G = (\mathbb{Z}, +)$

$$\int_G dg\, f(g) = c\sum_{n\in\mathbb{Z}} f(n)$$ same as finite. but not normalizable

3. $G = (\mathbb{R}^*_{>0}, \times)$ $\quad \int_G f(g)\, dg = c\int_0^{\infty} f(x)\frac{dx}{x}$

$$\forall a \in \mathbb{R}^*_{>0}: \int_0^{\infty} f(ax)\, d\mu(x) \overset{\text{inv.}}{=} \int_0^{\infty} f(x)\, d\mu(x)$$

$$\overset{x\to\frac{x}{a}}{=} \int_0^{\infty} f(x)\, d\mu\left(\frac{x}{a}\right) \qquad \Big\} \; d\mu(x) = \frac{dx}{x}$$

4. $G = U(1) = \{ z \in \mathbb{C} . \; |z| = 1 \}$

$$\int_{U(1)} \underline{\underline{d\mu(z)}}\, f(z) = \int_{U(1)} d\mu(z)\, f(z_0 z)$$

$$= \int_{U(1)} \underline{\underline{d\mu(z_0^{-1} z)}}\, f(z)$$

assume $d\mu(z) = \rho(z)\, dz$

$$d\mu(z_0^{-1} z) = \rho(z_0^{-1} z)\, d(z_0^{-1} z)$$

$$= z_0^{-1}\rho(z_0^{-1} z)\, dz = \rho(z)\, dz$$

This requires $\rho(z) \sim z^{-1}$. i.e. $d\mu(z) = c \cdot \frac{dz}{z}$

<u>Normalization</u> ? $\quad g(\phi) = f(z = e^{i\phi})$, $dz = iz\, d\phi$

$$c \int_{U(1)} \frac{dz}{z} = 1 \Rightarrow c \int_0^{2\pi} i\, d\phi = 1 \Rightarrow c = \frac{1}{2\pi i}$$

$$\int_{U(1)} d\mu(z) f(z) = \frac{1}{2\pi i} \int_{U(1)} f(z) \frac{dz}{z}$$

5. $G = GL(n, \mathbb{R})$ $\quad g \longmapsto g_0 g = g'$ $\quad \underline{\underline{g \in \mathbb{R}^{n^2}}}$

$$g'_{ij} = \sum_k (g_0)_{ik} g_{kj} \Rightarrow \frac{\partial g'_{ij}}{\partial g_{k\ell}} = (g_0)_{ik} \delta_{j\ell}$$

$$\frac{\partial g'_{ij}}{\partial g_{kj}} = (g_0)_{ik} \qquad \prod_{ij} d g'_{ij} \longmapsto \underline{\left| \frac{\partial (g'_{11} \cdots g'_{nn})}{\partial (g_{11} \cdots g_{nn})} \right|} \prod_{ij} d g_{ij}$$

$$\begin{array}{c|cccc} & 11\ 21 \ldots & ;\ 12, 22, \ldots & ;\ 13, 23 \ldots & \\ \hline 11,\ 21, \ldots & \begin{pmatrix} g_{011} & g_{012} & \\ & & \end{pmatrix} & 0 & 0 \\ & 0 & \begin{pmatrix} & \end{pmatrix} & 0 \\ & 0 & 0 & \begin{pmatrix} & \end{pmatrix} \\ & 0 & & \end{array}$$

$\det \oplus_i M_i = \prod_i \det M_i$

$$= |\det g_0|^n \prod_{ij} d g_{ij}$$

Haar measure $d\mu(g) = c \cdot |\det g|^{-n} \prod_{ij} d g_{ij}$

$$\int f(g_0 g)\, d\mu(g) = c \int f(g_0 g) |\det g|^{-n} \prod d g_{ij}$$

$$= c \int f(g) \, |\det g_0^{-1} g|^{-n} \prod d (g_0^{-1} g)_{ij}$$

$$= c \int f(g) \, |\det g|^{-n} \cancel{|\det g_0|^{n}} \underbrace{\cancel{|\det g_0|^{-n}} \prod d g_{ij}}_{\text{from above}}$$

$$= c \int f(g) \, |\det g|^{-n} \prod d g_{ij}$$

$$= \int f(g)\, d\mu(g)$$

Q: What about $GL(n, \mathbb{C})$?

6. $G = SU(2)$ $\qquad g \in SU(2)$

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$u(\phi, \theta, \psi) = U_z(\phi)\, U_x(\theta)\, U_z(\psi)$$

$$= e^{i\frac{\sigma_z}{2}\phi}\, e^{i\frac{\sigma_x}{2}\theta}\, e^{i\frac{\sigma_z}{2}\psi}$$

$$= \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\psi}{2}} & \\ & e^{-i\frac{\psi}{2}} \end{pmatrix}$$

$$\begin{cases} \alpha = e^{\frac{i}{2}(\phi+\psi)} \cos\frac{\theta}{2} \\ \beta = i e^{\frac{i}{2}(\phi-\psi)} \sin\frac{\theta}{2} \end{cases}$$

periodicity. $(\phi, \psi) \sim (\phi + 4\pi, \psi)$

$\sim (\phi, \psi + 4\pi) \sim (\phi + 2\pi, \psi + 2\pi)$

$$\begin{cases} \theta \in [0, \pi) \\ \phi \in [0, 2\pi) \\ \psi \in [0, 4\pi) \end{cases}$$

① $d\alpha\, d\bar{\alpha}\, d\beta\, d\bar{\beta} \to J\, dr\, d\psi\, d\phi\, d\theta$

$$J = \left| \frac{\partial(\alpha, \bar{\alpha}, \beta, \bar{\beta})}{\partial(r, \phi, \theta, \psi)} \right| = \frac{1}{2} r^3 \sin\theta \Big|_{r=1} = \frac{1}{2}\sin\theta$$

② $g \mapsto g_0 g$. $\quad |\det g_0| = 1$ $(\because SU(2))$ $\qquad$ note the parametrization ? the form will be different

$$\mu(g) = \underset{\frac{1}{16\pi^2}}{\cancel{c}} \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{4\pi} d\psi}_{2\pi \times 2 \times 4\pi \,=\, 16\pi^2}$$

7. Some functions defined on G are invariant under the action of a subgroup $H \subset G$. therefore depend only on G/H

Example. interger spins. formally on $SU(2)$, but actually follows $SO(3)$.

$s=\frac{1}{2}$: $U(\hat{n}, \theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\hat{\sigma}}$

$s=1$, $U(\hat{n}, \theta) = e^{-i\theta\hat{n}\cdot\hat{J}}$

$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$J_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$U(\hat{n}, 2\pi) = -1$ $\quad\quad$ $U(\hat{n}, 2\pi) = 1$

Earlier, we discussed

$$SU(2) \xrightarrow{\pi} SO(3)$$

$$u\vec{x}\cdot\vec{\sigma}u^{-1} = (\pi(u)\vec{x})\cdot\vec{\sigma}$$

$\pi(u) = \pi(-u)$

$\ker = \mathbb{Z}_2$

$\mathrm{im} = SO(3)$

1st iso. thm. $SO(3) \cong SU(2)/\mathbb{Z}_2$

On the group G. we have defined a Haar measure.

what about on G/H ($H = \mathbb{Z}_2$ here)?

$$d\mu_G \longrightarrow d\mu_{G/H} \quad ?$$

Integrate out H:

$$\pi_H(f)(g) = \int_H f(gh)\, d\mu_H(h)$$

left invariant measure on H

the $\pi_H(f)$ is just a function on G/H.

$$\pi_H(f)(g_1) = \pi_H f(g_2) \quad \text{if} \quad g_1 H = g_2 H. \quad \text{i.e.} \quad g_1 = g_2 h \ (h \in H)$$

Then we can define a measure $d\mu_{G/H}$ s.t.

$$\int_{G/H} \pi_H(f)(g)\, d\mu_{G/H}(gH) = \int_G f(g)\, d\mu_G(g)$$

$$d\mu_{SU(2)} = \frac{1}{16\pi^2}\, d\phi \, \sin\theta \, d\theta \, d\psi$$

After projection. identifies $\psi \sim \psi + 2\pi$

$$\pi_H(f)(\phi, \theta, \psi) = \frac{1}{2}\left[f(\phi, \theta, \psi) + f(\phi, \theta, \psi + 2\pi) \right]$$

$$d\mu_{SO(3)} = \frac{1}{8\pi^2}\, d\phi \, \sin\theta \, d\theta \, d\psi.$$

$\theta \in [0, \pi)$

$\phi \in [0, 2\pi)$

$\psi \in [0, 2\pi)$

Similarly. take H to be diagonals. $H = \left\{ \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix} \right\} \cong U(1)$

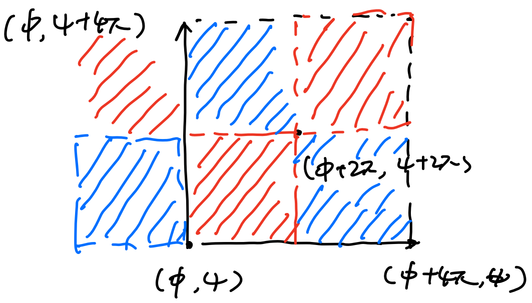
then $G/H = SU(2)/U(1) \cong S^2$ (orbit-stabilizer)

$$d\mu_{S^2} = \frac{1}{4\pi}\, d\phi \, \sin\theta \, d\theta \quad (\text{integrate over } \psi)$$

Some post lecture notes on Haar measure of SU(2):

1) fundamental domain for integration

$$(\phi, \psi) \sim (\phi + 4\pi, \psi) \sim (\phi, \psi + 4\pi) \sim (\phi + 2\pi, \psi + 2\pi)$$



reds are equivalent domains

blues are equivalent domains

choose one red + one blue for integration.

either $\phi \in [0, 2\pi)$

$\psi \in [0, 4\pi)$

or $\phi \in [0, 4\pi)$

$\psi \in [0, 2\pi)$