# 8. Representation theory

(Moore § 11)

## 8.1. Some motivation:

Group = symmetry ; Grp. rep : how symmetry acts on physical states.

structure of sym — structure of physics problem.

1 Earlier we discussed that physical states in QM are rays in Hilbert space. (normalized vectors $(\Psi, \Psi) = 1$

$R_\psi = \{ \psi' = \alpha \psi, \ \alpha \in U(1) \}$ )

symmetries are represented by

(unitary, linear), (antiunitary, antilinear)

operators in Hilbert space $\mathcal{H}$. $[\mathrm{Aut}(\mathcal{H})]$

( Wigner. 1931 ; Weinberg. QFT-I. 1995 )

$$\begin{cases} \text{unitary} & (U\Phi, U\Psi) = (\Phi, \Psi) \\ \text{linear} & U(\alpha\Phi + \beta\Psi) = \alpha U\Phi + \beta U\Psi \end{cases}$$

$$\begin{cases} \text{anti-uni}: & (U\Phi, U\Psi) = (\Psi, \Phi)^* \\ \text{anti lin}: & U(\alpha\Phi + \beta\Psi) = \alpha^* U\Phi + \beta^* U\Psi \end{cases}$$

$U^\dagger = U^{-1}$ for both.

① unitary: rotation, translation (includes identity)

② antiunitary: time-reversal $p \to -p$ $\quad U = 1 + i\delta \to$ hermition

$\hat{p} = -i\hbar \frac{\partial}{\partial x} \to$ complex conjugate.

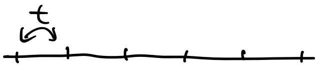
If the Hamiltonian $H$ has certain symmetry, represented by $S$, then

$$SHS^{-1} = H$$

For unitary $U$, $[H, U] = 0$ they have the same eigenstates.

$\Rightarrow$ simultaneous diagonalizable.

Consider <u>1D lattice</u>



$$H = -t \sum_i \left( |i\rangle\langle i+1| + |i+1\rangle\langle i| \right)$$

discrete translational sym: $T|i\rangle = |i+1\rangle$

In the basis $\{|i\rangle\}$, matrix rep of $H$ and $T$ are

$$H = \begin{pmatrix} 0 & t & - & - & t \\ t & 0 & t & & \vdots \\ & t & 0 & t & \vdots \\ \vdots & & t & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ t & - & - & & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & \\ 0 & 1 & \cdots & \cdots & - \\ - & - & \cdots & 1 & 0 \end{pmatrix}$$

easy to verify $[H, T] = 0$. $T$ unitary: $T^\dagger T = T T^\dagger = \mathbb{1}_N$

Both diagonalized by a fourier transform

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikj} |j\rangle$$

$$\begin{cases} H|k_i\rangle = -2t \cos k_i \, |k_i\rangle \\ T|k_i\rangle = e^{ik_i} |k_i\rangle \end{cases} \qquad k_i = \frac{2\pi}{N} i$$

cf. Noether theorem.

Spatial trans → momentum

time → energy

rot → angular

Each "<u>irrep</u>" of $T$ labels an eigenstate.

$\Leftrightarrow$ "$k$ is a good quantum number"

2 More generally, of a set of symmetries, $[H, u_i] = 0$

$\Rightarrow \exists S_i$. s.t

$$S_1 H S_1^{-1} = \begin{array}{|c|c|c|} \hline H_1 & 0 & 0 \\ \hline 0 & H_2 & 0 \\ \hline 0 & 0 & H_3 \\ \hline \end{array}$$ block-diagonal

$$S_2 H_1 S_2^{-1} = \begin{array}{c|c|c} H_{1,1} & & \\ \hline & H_{1,2} & \\ \hline & & \ddots \end{array}$$

symmetry sectors labeled by a set of different QNs

e.g. for Fermions $\underline{QN}$. = particle number

$\begin{cases} S \\ S_z \end{cases}$

$$\begin{array}{cccccc} |\uparrow\downarrow, -\rangle & |\uparrow, \downarrow\rangle & |\downarrow, \uparrow\rangle & |-, \uparrow\downarrow\rangle \end{array}$$

$$\begin{pmatrix} u & t & -t & 0 \\ t & 0 & 0 & t \\ -t & 0 & 0 & -t \\ 0 & t & -t & u \end{pmatrix} \oplus 0 \qquad |S_z\rangle$$

$$\downarrow$$

$$\begin{pmatrix} u & -\sqrt{2}t & 0 \\ -\sqrt{2}t & 0 & -\sqrt{2}t \\ 0 & -\sqrt{2}t & u \end{pmatrix} (0) \qquad |S, S_z\rangle$$

if a "probe" operator connects different symmetry sectors by changing some quantum number $\Rightarrow$ transition

"$\underline{\text{selection rules}}$"

## 8.2 Review of basic definitions

① $G \to GL(V)$

$V$ some vector space over field $K$

$GL(V)$ / $Aut(V)$ : invertible linear transformations $V \to V$.

② rep. of $G$. is a group homomorphism.

$$T: G \to GL(V)$$
$$g \mapsto T(g)$$

$(T, V)$ denotes the representation. or $T$ or $V$

$$T(g_1)T(g_2) = T(g_1 g_2) \quad \forall g_1, g_2 \in G.$$

$V$ is called the carrier space / representation space

dim $V$ is the dimension / degree of the representation.

In terms of group actions. rep. of $G$ is a $G$-action on a vector space that respects linearity

$$g\cdot(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 g\cdot v_1 + \alpha_2 g\cdot v_2 \qquad v_i \in V, \ \alpha_i \in K$$

Given an ordered basis of finite dim $V$.

$$\{\hat{e}_1, \cdots \hat{e}_n\} \Rightarrow GL(V) \cong GL(n, K)$$

$$T(g)\hat{e}_i = \sum_j M(g)_{ji}\, \hat{e}_j$$

$$T(g_1)[T(g_2)\hat{e}_i] = T(g_1)\sum_j M(g_2)_{ji}\hat{e}_j = \sum_j M(g_2)_{ji}\,(T(g_1)\hat{e}_j)$$
$$= \sum_j M(g_2)_{ji}\sum_k M(g_1)_{kj}\,\hat{e}_k = \sum_k [M(g_1)M(g_2)]_{ki}\,\hat{e}_k$$
$$T(g_1)T(g_2) = T(g_1g_2) \Leftrightarrow M(g_1)M(g_2) = M(g_1g_2)$$

( Reps are not unique.

$$\begin{pmatrix} M_1(g) & X(g) \\ 0 & M_2(g) \end{pmatrix}\begin{pmatrix} M_1(g') & X(g') \\ 0 & M_2(g') \end{pmatrix} = \begin{pmatrix} M_1(g)M_1(g') & M_1X + XM_2 \\ 0 & M_2(g)M_2(g') \end{pmatrix}$$ )

Later we will define reducible and irreducible reps.

Examples.

1. rep. of degree / dim 1.

$$T: G \to \mathbb{C}^*$$

for element of order $n$. $g^n = 1_G$

$T(g)^n = 1$ $T(g)$ are roots of 1

$\mathbb{Z}_3 \cong \mu_3 \cong A_3 = \langle g \rangle$ $T(g) = \omega = e^{i\frac{2\pi}{3}}$ / $e^{i\frac{4\pi}{3}}$

if take $T(g) = 1$ trivial representation (unit)

↑ trivial homo.

2. "regular representation" of a finite group.

(more to be discussed later, group algebra)

正则表示

Let $\dim V = |G| = n$. with an ordered basis set $\{\hat{e}_g\}$ $(g \in G)$

$$T(g_1)\cdot \hat{e}_{g_2} = \hat{e}_{g_1g_2}$$

| V | e | a | b | c |
|---|---|---|---|---|
| e | e | a | b | c |
| a | a | e | c | b |
| b | b | c | e | a |
| c | c | b | a | e |

$\langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$

$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$e = (0, 0)$

$a = (1, 0)$

$b = (0, 1)$

$c = (1, 1)$

$$T_{reg} : \mathbb{Z}_2 \times \mathbb{Z}_2 \to GL(V) \ (\dim V = 4)$$

$$V = \{\hat{e}_e, \hat{e}_a, \hat{e}_b, \hat{e}_c\}$$

$$T(e)\hat{e}_g = \hat{e}_g \qquad \underline{T(e) = \mathbb{1}_4} \qquad \begin{cases} \chi(T(e)) = \dim V \\ \qquad = 4 \\ \chi(T(g \neq e)) = 0 \end{cases}$$

$$\begin{cases} T(a)\hat{e}_e = \hat{e}_a \\ T(a)\hat{e}_a = \hat{e}_e \\ T(a)\hat{e}_b = \hat{e}_c \\ T(a)\hat{e}_c = \hat{e}_b \end{cases} \qquad T(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3. more generally. $G$ acts on set $X$

$$x \mapsto gx$$

Let $V$ be a vector space with basis $\{e_x\}$ $(x \in X)$

$$T(g) e_x = e_{gx}$$

permutation representation.

4. $G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ $\quad T : G \to GL(\mathbb{C})$

$$n \mapsto a^n \quad (a \in \mathbb{C}^*)$$

$$n_1 + n_2 \to a^{n_1} \cdot a^{n_2} = a^{n_1 + n_2}$$

5. $G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$. $\quad T : G \longrightarrow GL(2, k)$

$$n \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

6. $G = GL(n, k)$ $\rightsquigarrow$ one-dim. representation

$$T(g) := |\det g|^{\mu}$$

$$T(g_1 g_2) = |\det(g_1 g_2)|^{\mu} = |\det g_1|^{\mu} |\det g_2|^{\mu}$$
$$= T(g_1) T(g_2)$$

~~7~~. 1+1 dim Lorentz group

$$x^{0'} = \cosh\theta\, x^0 + \sinh\theta\, x^1$$

$$x^{1'} = \sinh\theta\, x^0 + \cosh\theta\, x^1$$

$$\begin{pmatrix} x^{0'} \\ x^{1'} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = B(\theta) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

$$\left( B(\theta) \in O(1, 1) = \{ A \mid A^T \eta A = \eta \},\ \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$B(\theta_1) \cdot B(\theta_2) = B(\theta_1 + \theta_2)$$

Examples <u>Direct sum, tensor product, and dual representations</u>

$(T_1, V_1)$ and $(T_2, V_2)$ are two reps of $G$

with $\dim V_1 = n$ and $\dim V_2 = m$, and basis

$\{v_1, \cdots, v_n\}$, $\{w_1, \dots, w_m\}$

① $V_1 \oplus V_2$. vector space of dim. $n+m$

with basis $\{(v_1, 0), (v_2, 0) \dots, (0, w_1), (0, w_2) \cdots\}$

rep on $V_1 \oplus V_2$: $\quad g\cdot(v, w) := (g\cdot v, g\cdot w)$ — G-action

$[(T_1 \oplus T_2)(g)](v \oplus w) := T_1(g) v \oplus T_2(g) w$. — rep.

mat. rep.

$$M_{T_1 \oplus T_2}(g) = \begin{array}{cc} n & m \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} M_{T_1}(g) & 0 \\ \hline 0 & M_{T_2}(g) \end{array}\right)} \end{array}$$

② $V_1 \otimes V_2$: vector space of dim $\underline{n \cdot m}$, basis

$\{v_i \otimes w_j : 1 \le i \le n,\ 1 \le j \le m.\}$

$$\left(\sum_i a_i v_i\right) \otimes \left(\sum_j b_j w_j\right) = \sum_{ij} a_i b_j\, v_i \otimes w_j$$

rep on $V_1 \otimes V_2$:

$$g\cdot(v \otimes w) = (g\cdot v) \otimes (g\cdot w)$$

$$[(T_1 \otimes T_2)(g)](v \otimes w) := T_1(g) v \otimes T_2(g) w$$

$$[(M_1 \otimes M_2)(g)]_{ik, jl} = (M_1(g))_{ij}\,(M_2(g))_{kl}$$

③ The dual vector space. $V^\vee$ (or $V^*$)

$\{$ linear maps: $V \to k \} := \mathrm{Hom}(V, k)$

with $v_i^\vee$. $\quad v_i^\vee(v_j) = \delta_{ij}$

$\dim V^\vee = \dim V = n.$

rep on $V^\vee$: $\quad (g \cdot v_i^\vee)(v_j) = v_i^\vee(g^{-1} \cdot v_j)$ (induced action on function space)

natural pairing: $\quad (g \cdot v_i^*)(g \cdot v_j) = v_i^*(g^{-1} \cdot g \cdot v_j)$

$$= v_i^*(v_j) = \delta_{ij}$$

$T(g): \; V \to V, \quad v \mapsto T(g)v$

$T^\vee(g): \; V^\vee \to V^\vee, \quad v^\vee \mapsto T^\vee(g) v^\vee$

$\underline{v_j = \sum_i M_{ij} v_i}$

$$\underline{v_i^\vee(v_j)} = \sum_k M^\vee(g)_{ki} v_k^\vee \cdot \left(\sum_l M(g)_{lj} v_l\right)$$

$$= \sum_{kl} M^\vee(g)_{ki} M(g)_{lj} \underbrace{v_k^\vee(v_l)}_{\to \delta_{kl}}$$

$$= \underline{\sum_l M^\vee(g)_{li} M(g)_{lj} = \delta_{ij}}$$

$$\Leftrightarrow \quad M^\vee(g) = [M(g^{-1})]^{tr} = M(g)^{tr,-1}$$

## 8.3 Equivalent representations and characters

**Definition.** Let $(T_1, V_1)$ and $(T_2, V_2)$ be two reps. of a group $G$. An intertwiner (intertwining map 交结映射) between these two reps is a linear transformation $A: V_1 \to V_2$ s.t. $\forall g \in G$, the following diagram commutes.

morphism / equivariant map

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ T_1(g)\downarrow & & \downarrow T_2(g) \\ V_1 & \xrightarrow{A} & V_2 \end{array}$$

i.e. $T_2(g)A = A \cdot T_1(g)$

$A$ is an equivariant linear map of $G$ spaces $V_1 \to V_2$

$\alpha A_1 + \beta A_2 \in \mathrm{Hom}_G$ $\quad \mathrm{Hom}_G(V_1, V_2)$ : vector space of all intertwiners.

**Definition.** Two reps $(T_1, V_1)$ and $(T_2, V_2)$ are equivalent $(T_1, V_1) \cong (T_2, V_2)$ if there is an intertwiner $A: V_1 \to V_2$ which is an isomorphism, that is

$$T_2(g) = A\, T_1(g) A^{-1} \quad (\forall g \in G)$$

For any finite-dimensional representation

$$T : G \to Aut(V)$$

of any group $G$. we can define the character of the representation $\chi_T$

$$\chi_T : G \to k$$

$$\chi_T(g) := Tr_V(T(g))$$

1. equivalent $\Leftrightarrow$ same character function

   and $\chi_T(h^{-1}gh) = \chi_T(g)$ "class function"

2. independent of basis choices

3. For above representations.

   a. $M_{T_1 \oplus T_2}(g) = \begin{pmatrix} M_{T_1}(g) & 0 \\ 0 & M_{T_2}(g) \end{pmatrix}$

   $$\chi_{T_1 \oplus T_2} = \chi_{T_1} + \chi_{T_2}$$

   b. $(M_1 \otimes M_2)(g)_{ik,jl} = (M_1(g))_{ij} (M_2(g))_{kl}$

   $$\chi_{T_1 \otimes T_2} = \sum_{ijkl} M^{\otimes}_{ik,jl} \delta_{ij} \delta_{kl} = \sum_{ik} [M_1]_{ii} [M_2]_{kk}$$
   $$= \chi_{T_1} \cdot \chi_{T_2}$$