Review of Group part

1. Definition of groups $(G.\ e.\ \underline{m}.\ \underline{I})$

① set G

② $e \in G$ $\quad e\cdot g = g \cdot e = g$.

③ $m:\ G \times G \to G$

④ $I:\ G \to G$

$G = \mathbb{Z} \cdot \mathbb{R} \cdot \mathbb{C}$ groups if $m = $"$+$"

not if $m = $"$\times$"

$\mathbb{R}^* = \mathbb{R} - \{0\}$. $\mathbb{C}^* = \mathbb{C} - \{0\}$

$\hookrightarrow$ $|G|$ order $\begin{cases} \text{finite} \\ \text{infinite} \end{cases}$ group

$\hookrightarrow$ $g_1 g_2 = g_2 \cdot g_1 \quad \forall g_1, g_2 \to$ abelian

$\not\to$ noabelian.

2. Direct product $H \times G$.

$(h_1, g_1) \cdot (h_2 \cdot g_2) = (h_1 \cdot h_2,\ g_1 \cdot g_2)$

↳ semidirect product $H \rtimes_\alpha G$. $h \in H$. $g \in G$.

$$(\underline{h_1}, \underline{g_1}) \cdot_\alpha (\underline{h_2}, \underline{g_2}) = (\underline{h\, \alpha_{g_1}(h_2)}, \underline{g_1 g_2})$$

$$\{R_1 | \tau_1\}\{R_2 | \tau_2\} \vec{r} = \{R_1 | \tau_1\} (R_2 \vec{r} + \vec{\tau}_2)$$

$$= R_1 R_2 \vec{r} + R_1 \vec{\tau}_2 + \vec{\tau}_1$$

$$(\tau_1, R_1)(\tau_2, R_2) = (R_1 \vec{\tau}_2 + \vec{\tau}_1, R_1 R_2)$$

↳ symmorphic space groups

3. subgroups $H \subset G$.

$\underline{m}: H \times H \to H$

$\underline{I}: H \to H$

G has trivial subgroups $\{e\}$ and G.

proper subgroup $H \neq G$

$\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ "+"

↳ $H \triangleleft G$: $gHg^{-1} = H$ $(\forall g \in G)$

↳ simple group, no nontrivial normal subgroup.

↳ centralizer $C_G(h) = \{g \in G.\ gh = hg\} \subset G$

$C_G(H) = \{g \in G.\ gh = hg,\ \forall h \in H\} \subset G$

normalizer $N_G(H) = \{g \in G.\ gHg^{-1} = H\}$

$C_G(H) \subset N_G(H)$

4. $GL(n, K)$

$\hookrightarrow SL(n, K)$

$\left.\begin{array}{ll} O(n, K), & SO(n, K) \\ U(n, K), & SU(n, K) \end{array}\right\}$ $\underline{\det}$

$A^T \underline{J} A = \underline{J}$ $\begin{cases} J_{p,q} = \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix} \\ \text{symplectic} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{cases}$

Homomorphism / isomorphism.

5. homomorphism. $\varphi : G \longrightarrow G'$

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \varphi \times \varphi \downarrow & & \downarrow \varphi \\ G' \times G' & \longrightarrow & G' \end{array}$$

$$\varphi(g_1) \underset{G'}{\cdot} \varphi(g_2) = \varphi(g_1 \underset{G}{\cdot} g_2)$$

$$\begin{cases} \varphi(e) = e' \\ \varphi(g^{-1}) = \varphi(g)^{-1} \end{cases}$$

ker / im : $\ker \varphi = \{ g \in G : \varphi(g) = 1_{G'} \}$

$\operatorname{im} \varphi = \varphi(G)$

① $\pi : SU(2) \longrightarrow SO(3)$

$$u\,\vec{x}\cdot\vec{\sigma}\cdot u^{+} := (\pi(u)\cdot\vec{x})\cdot\vec{\sigma}$$

$$\ker\pi = \{\pm 1\} \cong \mathbb{Z}_2$$

② $\Gamma : G \longrightarrow \underline{GL(V)}$ $\quad V$ some n-dim vector space over K

given basis

$$GL(V) \cong \underline{GL(n,K)}$$

<u>isomorphism</u> : homo. + (1-1 & onto)

1-1: $\ker\varphi = \{e\}$

onto: $\varphi(G) = G'$

$\varphi : G \to G$: $Aut(G)$

isomorphism defines an equivalence relation

$$\mu_N \cong \mathbb{Z}_N$$

<u>matrix-rep.</u> $\quad T : G \longrightarrow GL(n,K)$

$$T(g)\hat{e}_i = T(g)_{ji}\,\hat{e}_j$$

↳ equivalent rep $T \cong T'$, $\exists S$ s.t.

$$T'(g) = S\,T(g)\,S^{-1} \quad \forall g \in G.$$

more generally <u>conj. rep</u> $\quad \varphi_{1,2} : G \longrightarrow G'$

$$\varphi_2(g_1) = g_2\,\varphi_1(g_1)\,g_2^{-1}$$

6. define group action by homomorphism.

$$\Phi : G \longrightarrow S_X := \{ X \xrightarrow{f} X \}$$

set of permutations

$$g \longmapsto \phi(g, \cdot)$$

$$\Phi_g(x) = \phi(g, x) = g \cdot x$$

$$g_1 (g_2 \cdot x) = (g_1 g_2) x$$

↳ orbits. $Orb_G(x) = \{ g \cdot x . g \in G \}$

① defines equivalence relation

$$x \sim y . \quad y = g \cdot x$$

② orbits partition G.

$$O_G(x) = O_G(x') \text{ or}$$

$$O_G(x) \cap O_G(x') = \phi.$$

X/G set of orbits

↳ fixed points

$$Fix_X(g) = \{ x \in X : gx = x \} \subset X$$

⟶ stabilizer.

$$Stab_G(x) := \{ g \in G . g \cdot x = x \} \subset G.$$

$$(G^x)$$

Theorem. (Stab – orbit)

$$O_G(x) \xrightarrow{\cong} G/G^x$$

$$|O_G(x)| = [G : G^x]$$

$SO(3)$ acts on S^2. $\quad Orb_{SO(3)} = S^2$

$Stab_{SO(3)}(\hat{z}) \cong SO(2)$

$$S^2 \cong SO(3)/SO(2)$$

$SU(2)$ on $\mathbb{C}^2$. $\quad S^3 \cong SU(2)$

7. G – action on itself.

① H a subgroup, right action on G.

$gH = \{gh.\ h \in H\}$ left – cosets

$$|gH| = |H|$$

+ Lagrange. Finite G.

$$|G|/|H| = [G : H]$$

② action by conjugation.

orbits / conjugacy class

$$C(h) = \{ \underline{g} h g^{-1} \quad g \in G \}$$

Stab-orb. $|C(g)| = [G : C_G(g)]$

$\hookrightarrow \mathrm{Stab}_G(g)$

<u>centralizer</u>

Finite G. $|C(g)| = \dfrac{|G|}{|C_G(g)|}$

$+ \quad |G| = \Sigma |C(g)|$

$\Rightarrow \quad |G| = \displaystyle\sum_{\{C(g)\}} \frac{|G|}{|C_G(g)|}$ "class equation"

$\hookrightarrow$ ① $|G| = p^n \Rightarrow Z(G) \neq \{e\}$

② (Cauchy). $p \mid |G|$

$\Rightarrow \exists g \in G . \; g^p = 1$

<u>class function</u>.

function f on G.

$f(g g_0 g^{-1}) = f(g_0) \quad \forall g_0 . g . \in G.$

$\hookrightarrow$ meet rep.

$\chi_T(g) = \mathrm{Tr}\, T(g)$ character

$\hookrightarrow$ equivalent rep. $\varphi_1 . \varphi_2 \quad \exists g_2 .$ s.t.

$\varphi_2(g) = g_2 \varphi_1(g_1) g_2^{-1} \quad \forall g_1 \in G.$

mat. rep. $T_1 : G \to GL(n.k)$

$T_2 \ G \to GL(n.k)$

$\exists S \in GL(n.k).$ s.t.

$T_2(g) = S T_1(g) S^{-1} \qquad \forall g \in G.$

8. Morphisms of G spaces / equivariant map

$f : X \to X'$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

$f(\Phi(g,x)) = \phi'(g, f(x))$

$f(gx) = g \cdot f(x)$

9. The symmetric group S_n.

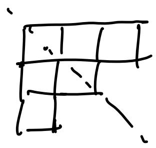
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = (1\,2\,4\,3)$$

① unique cycle decomposition. of $\phi \in S_n$

② r-cycles. are conjugate

↳ conjugacy classes labeled by partitions of n.

S_6. $\vec{\lambda} = \{3.2.1\}$



Young diagram.

$\mathrm{sgn}: S_n \to \mathbb{Z}_2$

$\phi \mapsto \mathrm{sgn}(\phi) = (-1)^{n-t}$ $\leftarrow$ len. of cycle decomposition.

$A_n \lhd S_n$ $\quad$ $\mathrm{sgn}(\phi \in A_n) = 1$

Why S_n?

finite G. of order n.

$\xrightarrow{\text{embed}} S_n$

$\cong$ some subgroup of S_n

$D_4 \cong S_4 \subset S_8$ $\quad$ $(|D_4| = 8)$

10. quotient groups

$N \lhd G$. then G/N has a natural group structure

$(g_1 N) \cdot (g_2 N) := (g_1 g_2) \cdot N$.

$\mu: G \to G/N$.

$g \mapsto gN$

$\ker \mu = N$.

1st. isomorphism theorem $\quad \mu: G \to G'$

$G/\ker\mu \cong \mathrm{im}\,\mu$.

11. exact sequence.

$$\to G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1}$$

$$\operatorname{im} f_{i-1} = \ker f_i$$

SES. $1 \xrightarrow[f_0]{} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow[f_3]{} 1$

① $\ker f_1 = \operatorname{im} f_0 = \{1\}$ $\quad f_1$ injective

② $\operatorname{im} f_2 = \ker f_3 = G_3$ $\quad f_2$ surjective.

$$1 \to N \to G \to Q \to 1$$

G is an extension of Q by N.

$$\begin{cases} N \cong H \trianglelefteq G \\ Q \cong G/N \end{cases}$$

↳ Central extension: $N \subset Z(G)$

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1$$