Recap.

1. Orbit-stabilizer theorem

G acts on X. $G^x \cong \mathrm{Stab}_G(x)$

then $\quad \mathcal{O}_G(x) \cong G/G^x$

$g\cdot x \mapsto g\cdot G^x$

only in the sense of sets, not a group isomorphism.

left: $\mathcal{O}_G(x)$ is only a set

right: G^x to be normal for G/G^x to be a group.

$hG^xh^{-1} = G^x \quad \forall h \in G$

$\Leftrightarrow hgh^{-1} = g' \Leftrightarrow hgh^{-1}x = x \Leftrightarrow gh^{-1}x = h^{-1}x \ \forall h$

$\Leftrightarrow g \in G^{h^{-1}x}$, $\underline{hx = x}$ $\forall h$. trivial orbit.

only the kernel of the action is normal.

Examples

① $SO(3)$ acts on S^2

$\mathrm{Stab}_{SO(3)}(\hat{z}) \cong SO(2)$, $\quad \mathrm{Orb}_{SO(3)}(\hat{z}) \cong SO(3)/SO(2)$
$\cong S^2$

② $SU(2)$ acts on $\mathbb{C}^2/S^3$

$\mathrm{Stab}_{SU(2)}(\hat{n}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\quad \mathrm{Orb}_{SU(2)}(\hat{n}) \cong SU(2)/\{1\}$
$\cong S^3$

2. Group acts on group:

① Conjugacy classes and centralizers

$$O_G(h) = \{ghg^{-1} . g \in G\} \equiv C(h)$$

$$Stab_G(h) = \{g \in G \mid ghg^{-1} = h\} \equiv \underline{C_G(h)}$$

subgroup

② conjugates all subgroups

$$O_G(H) = \{gHg^{-1} . g \in G\}$$

$$Stab_G(H) = \{g \in G \mid gHg^{-1} = H\} =: N_G(H)$$

③ right multi. of $H \subset G$ on G ?

$$O_H(g) = \{gh . h \in H\} = gH$$

$$Stab_H(g) = \{h \in H \mid gh = g\} = \{e\}$$

$$|gH| = |H/\{e\}| = |H|$$

3. class eqn.

$$|C(g)| = \frac{|G|}{|C_G(g)|}$$

$$|G| = \sum \frac{|G|}{|C_G(g)|} = |Z(G)| + \sum_{g \notin Z(G)} \frac{|G|}{|C_G(g)|}$$

7.4 Example applications of the stabilizer concept

Stabilizer code $S \subset P_n$ $\quad S = \langle g_1 g_2 \cdots \rangle$

$$V_S = \{ |\psi\rangle : s|\psi\rangle = |\psi\rangle, \ \forall s \in S \} \quad \text{code space}$$

Motivation: quantum errors:

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$X|1\rangle = |0\rangle$ "bit-flip"

$Z|0\rangle = |0\rangle$ "phase-flip"

$Z|1\rangle = -|1\rangle$

$$|0\rangle \xrightarrow{P_x \ll 1} \begin{cases} |0\rangle & 1-p & \checkmark \\ |1\rangle & p & \times \quad 10^{-2} \end{cases}$$

$|1\rangle$

$$|0\rangle \xrightarrow{CNOT} |000\rangle \longrightarrow \begin{cases} |000\rangle & (1-p)^3 & \\ |100\rangle & p(1-p)^2 & 10^{-2} \\ |110\rangle & p^2(1-p) & 10^{-4} \\ |111\rangle & p^3 & 10^{-6} \end{cases}$$

check by hand if error or logical operation

$I, \ X_i, \ X_i X_j, \ X_1 X_2 X_3$. 8 possibilities

$1 + 3 + 3 + 1$ for bit-flip

① ~exponential in n qubits

② collapses states. measure X_1? $X_1|000\rangle \to |100\rangle$

$\Rightarrow$ need a systematic way for error detection & correction

① Set up

Consider the Pauli group $P^n = (P_1)^{\otimes n}$

$P_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$

and its group action on the vector space

spanned by n-qubit states.

$G = P_n \qquad X = (\mathbb{C}^2)^{\otimes n}$

② Stabilizer subgroups and code spaces

Consider $S \subset P^n$ a subgroup.

Define $V_S = \{ |\varphi\rangle : s|\varphi\rangle = |\varphi\rangle, \ \forall s \in S \}$

$\begin{cases} V_S \text{ is the vector space stabilized by } S \quad \text{(Code space)} \\ S \text{ is the stabilizer of space } V_S. \end{cases}$

For V_S to be nontrivial.

1. $\forall s_1, s_2 \in S \qquad s_1 s_2 = s_2 s_1 \qquad S$ abelian

$s_1 s_2 |\varphi\rangle = s_1 |\varphi\rangle = |\varphi\rangle$

$s_2 s_1$

2. $\alpha I \in S. \qquad \alpha I |\varphi\rangle = |\varphi\rangle \quad \alpha = 1$

i.e. $-I, \pm iI \notin S$

$(-I|\varphi\rangle = |\varphi\rangle \Rightarrow |\varphi\rangle = \vec{0})$

$\dim V_S = 2^{n-r} \qquad n$: # physical qubits $\quad r$: independent generators

③ <u>normalizer</u> : logical operators

$\forall P \in P_n$. represents a logical operation on V_s

$S(P|\varphi\rangle) = P(P^{-1} S P)|\psi\rangle = P s' |\psi\rangle \equiv P|\psi\rangle \quad \forall s, \forall |\psi\rangle$

$\Rightarrow s' = P^{-1} s P \in S$.

$\Rightarrow P \in N(S) = \{ P \in P_n . | PSP^{-1} = S \}$

$P \in S$. acts trivially $P|\psi\rangle = |\psi\rangle$

$P \in N(S) \setminus S$: nontrivial logical operators

distance $d = \min\limits_{P \in N(S) \setminus S} wt(P)$. $\quad wt P = |\{ i : P_i \neq I \}|$

$P = Z_1 Z_2 I_3 \quad wt = 2$

$P_1, P_2 \in N(S)$. then

$P_1|\varphi\rangle = P_2|\varphi\rangle$ if $P_1^{-1} P_2 \in S$

Then the cosets $N(S)/S$ represents <u>distinct</u> logical operations on V_s

Standard notation for stabilizer codes:

$[n, \; k = n - r, \; d]$

physical $\quad$ logical

Example $n=3$ bit-flip code.

ignore phase for now

Now consider $S = \langle Z_1 Z_2, Z_2 Z_3 \rangle = \{I, Z_1 Z_2, Z_2 Z_3, Z_1 Z_3\}$

$\subset P_3$

$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$Z_1 Z_2$: $|000\rangle, |001\rangle, |110\rangle, |111\rangle$

$Z_2 Z_3$: $|000\rangle, |100\rangle, |011\rangle, |111\rangle$

$V_S = \text{span}\{|000\rangle, |111\rangle\}$ $\quad \dim V_S = 2^{n-r} = 2^{3-2} = 2$ $\quad$ 1 logical qubit

$\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$

$N(S) = \{P \in P_3 \mid [P, Z_1 Z_2] = [P, Z_2 Z_3] = 0\}$ $\quad \forall Z_i \in N(S)$

$\Rightarrow d = 1$

Code: $[n, k, d] = [3, 1, 1]$

if an operator O only contains Z, then $O \in N(S)$

$\{I, Z_i, Z_i Z_j, Z_1 Z_2 Z_3\}$ 8 in total.

or even X or Ys on i, j then $[O, Z_i Z_j]$ $\quad XZ = -ZX$, $YZ = -ZY$

$\{X_1 X_2 X_3, X_1 X_2 Y_3, \cdots\}$ 8 in total

we only care about $N(S)/S = \{S, \bar{X}S, \bar{Z}S, \bar{X}\bar{Z}S\} \cong V$

$\bar{X} = X_1 X_2 X_3$, $\bar{Z} = Z_1 Z_2 Z_3$ $\quad \{\bar{X}, \bar{Z}\} = 0$ $\quad \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\begin{cases} \bar{X}|\bar{0}\rangle = |\bar{1}\rangle & \bar{Z}|\bar{0}\rangle = |\bar{0}\rangle \\ \bar{X}|\bar{1}\rangle = |\bar{0}\rangle & \bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle \end{cases}$$

same as a single qubit

Consider error set: $\langle X_1, X_2, X_3 \rangle$ bit-flips

$\{X_1, Z_1\} = 0$

If E anticommutes with $s \in S$

$s|\varphi\rangle = |\varphi\rangle$

$\underline{sE|\varphi\rangle} = -Es|\varphi\rangle = -|\varphi\rangle \qquad E|\varphi\rangle \in \underline{U_s^{\perp}}$

$\Rightarrow$ detectable

If E commutes $\forall s \in S \qquad (E \in N(S) - S)$

$N(S) = \{ g \in P^n,\ gs = sg,\ \forall s \in S \}$

$s\underline{E|\varphi\rangle} = Es|\varphi\rangle = \underline{E|\varphi\rangle} \qquad E|\varphi\rangle \in \underline{U_s}$

$\Rightarrow$ undetectable

in particular, phase-flips Z_i, $(Z_i Z_j,\ Z_1 Z_2 Z_3 \cdots)$

<u>Syndrome measurement</u>

$M_1 = Z_1 Z_2, \qquad M_2 = Z_2 Z_3$

"syndrome" $(m_1, m_2) = (\langle\psi|M_1|\psi\rangle, \langle\psi|M_2|\psi\rangle) = (\pm 1, \pm 1)$

M_1	M_2	error
1	1	0
1	-1	flip 3
-1	1	flip 1
-1	-1	flip 2

$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$

measure with M_1:

$M_1|000\rangle = |000\rangle$

$M_1|111\rangle = |111\rangle$

…

for phase flip: Z_1: $\alpha|000\rangle + \beta|111\rangle \to \alpha|000\rangle - \beta|111\rangle$

syndrom $(1, 1)$ nondetectable.

formally, $[n, k, d]$ code can detect up to $d-1$ general errors.

$$d = \min_{P \in N(S) \setminus S} wt(P)$$

① for $wt(E) \geq d$. E can be in $N(S)\setminus S$. then $ESE^{-1} = s' \in S$

$s(E|\psi\rangle) = E(E^{-1}sE)|\psi\rangle = Es'|\psi\rangle = E|\psi\rangle$ $\quad E|\psi\rangle \in V_s$

undectable.

② $wt(P) < d$. $E \in S$ or $E \notin N(S)$

$E \in S$. nothing changes

$E \notin N(S)$ $\quad ESE^{-1} = \pm s$. $\quad s \in S$.

$\exists s$. s.t. $\{E, s\} = 0$. detectable.

So, $[3, 1, 1]$ can detect 0-bit arbitrary errors

1-bit bit-flip.

smallest stabilizer code for 1 bit:

$[4, 2, 2]$

(at least some X and Z on the same bit)