Recap.

1. $N \triangleleft G$. $\quad gNg^{-1} = N \quad \forall g \in G$.

   c.f. center $Z(G)$: $gzg^{-1} = z \quad \forall z \in Z(G)$, $g \in G$.

   ① trivial N: $\{1\}$ G.

   G has only trivial N : "simple group"

   (Lagrange) μ_p simple

   ② any hom $\mu : G \to G'$

   $K = \ker \mu \triangleleft G$

2. Quotient groups.

   ~~cosets~~ $G/N = \{gN,\ g \in G\}$

   $(g_1 N)(g_2 N) := (g_1 g_2) N$

   $$\hookrightarrow = g_1 g_2 \underbrace{g_2^{-1} N g_2}_{=N} N$$

   $$= g_1 g_2 N$$

3. Given N. construct hom

   $\phi : G \to G/N$ $\quad$ surjective.

   $g \mapsto gN$

   $\ker \phi = N$

4.
$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[d, hook, "\phi"'] & \mu(G) \subset G' \\ G/K \arrow[ur, "\text{inj.}"'] & \end{tikzcd}$$

   ϕ surj.

   1st iso. thm. $\quad G/K \cong \mu(G)$: $gK \mapsto \mu(g)$

6.4 Quotient groups and short exact sequences

$$\cdots G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} \cdots$$

$$\operatorname{im} f_{i-1} = \ker f_i$$

short exact sequence. $1 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} 1$

$f_0 : \{1\} \to \{1_{G_1}\}$ $\operatorname{im} f_0 = \{1\}$, $f_3 : G_3 \to \{1\}$ $\ker f_3 = G_3$

① G_1: $\ker f_1 = \{1_{G_1}\}$ injective

② G_2: $\operatorname{im} f_1 = \ker f_2$

③ G_3: $\operatorname{im} f_2 = G_3$ surjective

Now consider a homomorphism $\mu : G \to G'$

$K = \ker \mu$.

We have

inclusion map

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{\mu} \operatorname{im}\mu \longrightarrow 1$$

$\cong G/K$

Exactness check:

① K. $\ker i = \{1_G\}$ ✓

② G: $\ker \mu = \operatorname{im} i = K$ ✓

③ $\operatorname{im}\mu$: $\ker(\operatorname{im}\mu \to 1) = \operatorname{im}\mu$ ✓

1st isomorphism theorem $\Rightarrow$

$$\boxed{1 \longrightarrow K \longrightarrow G \longrightarrow G/K \longrightarrow 1}$$

<u>Remarks</u>.

1. If we have SES.

$$1 \longrightarrow N \xrightarrow{f_1} G \xrightarrow{f_2} Q \longrightarrow 1$$

then $N \cong H \trianglelefteq G$ (it is iso. to the kernel of homomorphism $G \to Q$)

We sometimes write Q as $G/f_1(N)$

where $f_1 : N \xrightarrow{f} G$ is an injective homomorphism.

"G is an extension of Q by N"

Example.

1. $1 \longrightarrow G_1 \longrightarrow G_1 \times G_2 \longrightarrow G_2 \longrightarrow 1$

$\quad (G_2) \qquad\qquad\qquad (G_1)$

$$\mu: G_1 \times G_2 \longrightarrow G_2 \qquad \left(\begin{matrix} g_1 \in G_1 \\ g_2 \in G_2 \end{matrix}\right)$$
$$(g_1, g_2) \longmapsto g_2$$

2. $\varphi: \mu_4 \longrightarrow \mu_2 \qquad (\mathbb{Z}_4 \longrightarrow \mathbb{Z}_2)$

$$w \longmapsto w^2 \qquad w = e^{i\frac{2\pi}{4}}$$

$$\ker \varphi = \{\pm 1\} \cong \mathbb{Z}_2$$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

in general $1 \longrightarrow \mathbb{Z}_n \longrightarrow \mathbb{Z}_{n^2} \longrightarrow \mathbb{Z}_n \longrightarrow 1$

$$\left(\varphi: \mu_{n^2} \to \mu_n \right)$$
$$z \longmapsto z^n$$

3. $\det: \quad O(n) \longrightarrow \mathbb{Z}_2 \qquad AA^T = \mathbb{1} \Rightarrow \det = \pm 1$

$$M \longmapsto \det(M)$$

$$\ker(\det) = SO(n)$$

$$1 \to SO(n) \longrightarrow O(n) \longrightarrow \mathbb{Z}_2 \to 1$$

Does it work?

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow O(n) \longrightarrow SO(n) \to 1$$

$\mu : O(n) \longrightarrow SO(n)$?

for some axis $\hat{n}$: $R_\theta \in SO(n)$. then $\sigma R_\theta \in O(n) \setminus SO(n) = PSO(n)$ σ: reflection

$\mu(R_\theta) = \mu(\sigma R_\theta) = R_\theta$

$\mu(\underbrace{\sigma R_\theta \sigma}_{R_{-\theta}} R_\phi) = \mu(R_{\phi-\theta}) = R_{\phi-\theta} \neq \mu(\sigma R_\theta)\mu(\sigma R_\phi)$

4. det $\quad U(n) \longrightarrow U(1)$

$\quad u \longmapsto \det u = \lambda$ $\quad$ ker ? $\quad SU(n)$

$1 \longrightarrow SU(n) \longrightarrow U(n) \longrightarrow U(1) \longrightarrow 1$

5. $\pi : SU(2) \longrightarrow SO(3)$

$u \vec{x}\cdot\vec{\sigma}\, u^+ := (\pi(u)\vec{x})\cdot\vec{\sigma}$

$u \in \ker \pi$. $\quad u \vec{x}\cdot\vec{\sigma}\, u^+ = \vec{x}\cdot\vec{\sigma}$ $\quad u = \lambda \mathbb{1}$

$\lambda = \pm 1$

$\pi(u) = \pi(-u)$

$\Rightarrow \ker\pi \cong \mathbb{Z}_2$

$SU(2)/\mathbb{Z}_2 \cong SO(3)$

rot. of classical vectors

$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU(2) \longrightarrow SO(3) \longrightarrow 1$

quantum spin states

abelian . and $\mathbb{Z}_2 \subset Z(SU(2))$

phase ambiguity / different projective reps

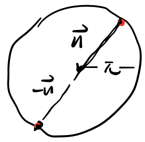
Analogy in 2D?

linear ← projective

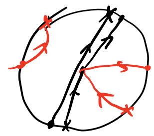
$$1 \to \_\_ \to \_\_ \to SO(2) \to 1$$

$\cong U(1)$

$\pi_1(SO(2)) \cong \mathbb{Z}.$



$\sim \mathbb{RP}^3$



$G/\mathbb{Z} \cong SO(2)$. $G = \mathbb{R}.$

$$1 \to \mathbb{Z} \to \mathbb{R} \to SO(2) \to 1$$

$U_s(\theta) = e^{i\theta s}$ arbitrary phases anyons

These are examples of:

<u>Definition</u> (central extension)

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

1. A is abelian.
2. $A \subset Z(E)$ $\quad i(a)\, b = b\, i(a)$ $(\forall a \in A, \forall b \in E)$

$$1 \to N \to G^{\text{Quantum}} \to G^{\text{classical}} \to 1$$

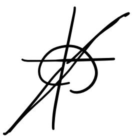
Motivation for such extensions in QM §11.1, §15.1 & 2

In QM, we talk about $\mathcal{H}\setminus\{0\}$ but distinct states are rep'ed by a set of vectors:

$$|\psi_1\rangle \sim |\psi_2\rangle \quad \text{iff} \quad |\psi_1\rangle = \lambda|\psi_2\rangle,\ \lambda \in \mathbb{C}^*$$

"rays" $\underline{\psi} = \{e^{i\alpha}\psi : \alpha \in \mathbb{R}\}$

This is actually the projective complex plane. $(z_1, z_2, z_3) \sim \lambda(z_1, z_2, z_3)$, $(\lambda \in \mathbb{C}^*)$ $\in \mathbb{CP}^3$



represented by $[z_1 : z_2 : z_3]$

This is not a linear space:

$[1:0:0] \overset{?}{+} [0:1:0]$ cannot be defined.

no zero vector $[0:0:0]$

uniquely defined by the density matrix

$$\rho_\varphi = \frac{|\varphi\rangle\langle\varphi|}{\langle\varphi|\varphi\rangle} \qquad (\rho^2 = \rho)$$

projective Hilbert space $P\mathcal{H} := (\mathcal{H}\setminus\{0\})/\sim$

Consider symmetry operations on $P\mathcal{H}$.

The overlap. $o(\rho_1, \rho_2) = \mathrm{Tr}(\rho_1\rho_2) = \dfrac{|\langle\varphi_1|\varphi_2\rangle|^2}{\|\varphi_1\|^2\|\varphi_2\|^2}$

should be conserved.

But for all kinds ~~of~~ reasons we want to work on <u>liner spaces</u> $\mathcal{H}$. <u>Aut $(\mathcal{H})$</u>

[<u>Wigner's theorem</u>: sym. operations are unitary or antiunitary

$$\begin{cases} \langle U\varphi, U\phi\rangle = \langle \varphi, \phi\rangle \\ \langle A\varphi, A\phi\rangle = \overline{\langle \varphi, \phi\rangle} = \langle \phi, \varphi\rangle \end{cases}$$

$(Ai = -iA \, , \, A = UK)$
↪ cplx. conj]

Now consider unitary symmetry operations on states.

one only needs

$$U(g_1)U(g_2)\varphi = \underline{\alpha(g_1, g_2)}\, U(g_1 g_2)\varphi$$

$\alpha: G\times G \to U(1)$

Not quite a group representation. projective rep.

(if $\tilde{U}(g) = b(g)\,U(g)$

$\Rightarrow b(g_1)\, b(g_2) = f(g_1, g_2)\, b(g_1 g_2) \quad \forall g_1, g_2$)

reduce back to rep.

rotation of spins:

classical $\quad$ quantum

$SO(3) \longrightarrow SU(2)$

Euler angles (2×2)

$$R(\phi, \theta, \psi) \to e^{i\frac{\phi}{2}\sigma^3} e^{i\frac{\theta}{2}\sigma^1} e^{i\frac{\psi}{2}\sigma^3}$$

$R(2\pi, 0, 0) = 1 \quad \to \quad \pm 1$ for fermion / boson

a $\mathbb{Z}_2$ phase

The central extension of G by A

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

is classified by the 2-cohomology group $H^2(G, A)$

6. finite Heisenberg group.

$$P = \begin{pmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix} \qquad Q = \begin{pmatrix} \omega & & & 0 \\ & \omega^2 & & \\ & & \omega^2 & \\ 0 & & & 1 \end{pmatrix}$$

$$\omega = e^{i\frac{2\pi}{4}}$$

$$\underline{\underline{QP = \omega PQ}} \quad \leftarrow$$ Weyl relation of canonical commutation relation.

Some background.

$[q, p] = i\hbar \quad (\hbar = 1)$

$qp - pq = i \qquad p, q$. acts on $f(\mathbb{R})$

$$\Rightarrow \quad A = e^{i\xi p} \qquad B = e^{i\eta q} \qquad (\text{Weyl.})$$

$$AB = e^{i\xi p} \cdot e^{i\eta q}$$

$$= e^{i(\xi p + \eta q) + \frac{1}{2}[i\xi p, i\eta q]}$$

$$BA = e^{i(\xi p + \eta q) + \frac{1}{2}[i\eta q, i\xi p]}$$

$$e^X e^Y = e^Z$$

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\left([X, [X, Y]] - [Y, [X, Y]]\right) + \cdots$$

$$\Rightarrow \quad AB = e^{i\xi\eta} BA \equiv \omega BA \qquad (A, B : n\times n \text{ mats.})$$

$$\det(AB) = \omega^n \det(BA) \Rightarrow \underline{\omega^n = 1}$$

$$\begin{cases} A^k B = \omega^k B A^k \\ A B^l = \omega^l B^l A \end{cases} \rightarrow A^k B^l = \omega^{kl} B^l A^k$$

$k=n \cdot l=1 \Rightarrow \underline{A}^n B = \cancel{\omega^{n}}^{1} B A^n \overset{?}{\Rightarrow} A^n = \mathbb{1}$

similary $B^n = \mathbb{1}$.

A general element in $\underline{\mathrm{Heis}_N}$ has the form

$$\omega^a P^b Q^c$$

$$(\omega^{a_1} P^{b_1} Q^{c_1}) \cdot (\omega^{a_2} P^{b_2} Q^{c_2}) = \omega^{a_3} P^{b_3} Q^{c_3}$$

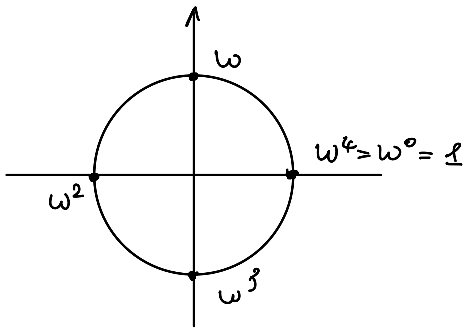
$$\begin{cases} a_3 = a_1 + a_2 + c_1 b_2 \\ b_3 = b_1 + b_2 \\ c_3 = c_1 + c_2 \end{cases}$$

$$\pi : \mathrm{Heis}_N \longrightarrow \mathbb{Z}_N \times \mathbb{Z}_N$$

$$\omega^a P^b Q^c \longmapsto (b \bmod N, c \bmod N)$$

$$\ker(\pi) = \{ \omega^a \cancel{P^{rN}}^{\mathbb{1}} \cancel{Q^{sN}}^{\mathbb{1}} \} \cong \mathbb{Z}_N$$

$$1 \longrightarrow \mathbb{Z}_N \longrightarrow \mathrm{Heis}_N \longrightarrow \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \underline{1}$$



$(P \cdot \Psi)(\omega^k) := \Psi(\omega^{k-1})$ translation

$(Q\Psi)(\omega^k) := \omega^k \Psi(\omega^k)$ position operator

$$\begin{cases} (QP)\Psi(\omega^k) = \omega^k P\Psi(\omega^k) = \omega^k \Psi(\omega^{k-1}) \\ (PQ)\Psi(\omega^k) = Q\Psi(\omega^{k-1}) = \omega^{k-1}\Psi(\omega^{k-1}) \end{cases}$$

$\Rightarrow QP = \omega PQ$

"Fourier transform" to "plane waves"

take $\Psi_j(\omega^k) = \omega^{-jk}$

$$P\Psi_j(\omega^k) = \Psi_j(\omega^{k-1}) = \omega^j \Psi_j(\omega^k)$$

$$Q\Psi_j(\omega^k) = \omega^k \Psi_j(\omega^k) = \Psi_j(\omega^{k-1})$$

$P \leftrightarrow Q$ switch places.

$N \to \infty$: $\mathbb{Z}_N \to U(1)$

7. More on group actions

7.1. Some defs and S-O theorem

Recall that the group action of G on a set X:

$$\phi : G \times X \to X$$

① left action. $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$$

(right action $(x \cdot g_2) \cdot g_1 = x \cdot (g_2 g_1)$)

② $\phi(1_G, x) = x$

mention different forms of L & R actions and induced actions on $\mathcal{F}[X \to Y]$

A G-action is: see Moore's note

① effective: $\forall g \neq 1, \exists x$. s.t. $gx \neq x$

(ineffective $\exists g \neq 1, \forall x$. s.t. $gx = x$)

② transitive: $\forall x, y \in X$. $\exists g$. s.t. $\underline{y = g \cdot x}$

there is only one orbit

③ free: $\underline{\forall g \neq 1}$. $\underline{\forall x}$. $\underline{g \cdot x \neq x}$

Definitions.

1. isotropy group (stabilizer group

$$\mathrm{Stab}_G(x) := \{ g \in G \ . \ g \cdot x = x \} \subset G$$

$(\equiv G^x)$

$$\left(\begin{array}{c} g_1 \, . \, g_2 \in G^x \\ g_1 \cdot x = x \quad g_2 \cdot (g_1 \cdot x) = g_2 x = x \\ g_2 g_1 \in G^x \end{array} \right)$$

If the group action of G is <u>free</u>

$\iff G^x = \{1\} \quad \forall x \in X.$

2. If $\exists g (\in G^x) \neq 1 \quad g \cdot x = x$. x is called a fixed point.

$$(X^g \equiv) \, \mathrm{Fix}_X(g) = \{ x \in X : g \cdot x = x \} \subset X$$

is the fixed point set of g.

free $\iff X^g = \phi \quad (g \neq 1)$

3. $O_G(x) = \{ gx . \ \forall g \in G \}$