Recap:

1. Conjugacy $h' = g h g^{-1}$ $g \in G$.

$C(h) = \{ g h g^{-1}, \forall g \in G \}$

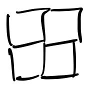
orbits of G acts on G.

2. $H \subset G$ a subgroup. then gHg^{-1} a subgroup.

3. S_n : same cycle structure $\sim$

$\Rightarrow$ Young diagrams

4 1-cycles $(1)(2)(3)(4)$



$(12)(34)$

$\vdots$

5 conjugacy classes

4. class functions : f on G.

$f(hgh^{-1}) = f(g)$

character $\chi_T(g) = \mathrm{Tr}\ T(g)$ for matrix rep.

canonical rep. of S_n $[A(\phi)]_{ji} = \delta_{j,\phi(i)}$ then

$\mathrm{Tr}\ A(\phi) = \#$ fixed points ✓

5. conjugate hom. $\varphi_1(g) = g_0\, \varphi_2(g)\, g_0^{-1}$ $\forall g \in G$.

equivalent rep. $T_2(g) = S\, T_1(g) S^{-1}$ $\forall g \in G$.

– 6.3. Normal subgroups & Quotient groups

Defition A subgroup $N \subset G$ is called a normal subgroup or an invariant subgroup if

$$gNg^{-1} = N \quad \forall g \in G.$$

denoted $N \triangleleft G$. (self-conjugate subgroups)

* NB. it doesn't mean $gng^{-1} = n \quad \forall n \in N$!

Suppose a subgroup Z satisfies

$$gzg^{-1} = z \quad \forall z \in Z \quad \forall g \in G.$$

$$Z(G) := \{ z \in G \mid zg = gz. \ \forall g \in G \}$$

$Z(G)$ is an abelian normal subgroup of G.

$Z(G)$ is the center of G.

Examples.

1. G is abelian. all subgroups are normal.

$$ghg^{-1} = (gg^{-1})h = h \quad \forall h \in G.$$

$C(h)$?

2. The kernel of a homomorphism

$$\phi : G \longrightarrow G'$$

is a normal subgroup.

$k \in \ker(\phi)$. $\phi(k) = 1_{G'}$

$\phi(gkg^{-1}) = \phi(g)\cancel{\phi(k)}\phi(g^{-1}) = \phi(g)\phi(g)^{-1} = 1 \quad (\forall g \in G)$

$\Rightarrow gkg^{-1} \in \ker(\phi)$

$\Rightarrow \ker\phi \triangleleft G$

<u>Theorem</u>. If $N \triangleleft G$. then the set of left cosets $G/N = \{gN, g \in G\}$. has a <u>natural</u> group structure with group multiplication defined as

o $(g_1N)\cdot(g_2N) := (g_1g_2)N$

We call the groups of the form G/N <u>quotient groups</u>. (factor groups)

$g_1N \cdot g_2N = g_1(g_2g_2^{-1})Ng_2N$

$= g_1g_2\,(\underline{g_2^{-1}Ng_2})N$ $= N$

$= g_1g_2N$

Corollary. If $N \triangleleft G$. then the natural map

$$\phi : G \longrightarrow G/N$$
$$g \longmapsto gN$$

is a surjective homomorphism. $\ker\phi = N$

① $\phi(g_1)\phi(g_2) = g_1N \cdot g_2N = g_1g_2N = \phi(g_1g_2)$

② $\phi(e) = e \cdot N = N = e_{G/N}$ $\quad (NN = N, \quad gN \cdot N = N \cdot gN = gN)$

$g \in \ker\phi \quad \phi(g) = gN = N \iff g \in N$

Every normal subgroup is the kernel of some homomorphism.

Example.

1 $n\mathbb{Z} := \langle n \rangle \triangleleft \mathbb{Z}$

$= \{ \cdots - 2n, -n, 0, n, 2n, \cdots \}$

$\mathbb{Z}/n\mathbb{Z} := \{ i + n\mathbb{Z},\ 0 \le i \le n-1 \}$

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$$
$$i \longmapsto i + n\mathbb{Z}$$

$\ker\phi = n\mathbb{Z}$

$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$

quotient groups are not subgroups

special cases . e.g

$$\left(\begin{array}{c} \mathbb{Z}_2 \lhd \mathbb{Z}_4 \\ \mathbb{Z}_4/\mathbb{Z}_2 \cong \mathbb{Z}_2 \end{array} \right)$$

2. $A_3 \lhd S_3$ $\quad \phi: S_3 \to \mathbb{Z}_2 \quad \ker(\phi) = A_3$

[HW] $H \subset G$. $[G:H]=2 \Rightarrow H \lhd G$ discussed earlier

3. $D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle \quad |D_4| = 8 = 2^3$

$D_4 = \langle e, a, a^2, a^3, b, ab, a^2b, a^3b \rangle$

$$\left(\begin{aligned} \underline{ba^n} = b^{-1}a^n = (ab)^{-1}a^{n+1} &= aba^{n+1} \\ &= a^2ba^{n+2} \end{aligned} \right)$$

non-trivial normal subgroups.

① $\{e, b, a^2b, a^2\} = N_1$

$a\underline{ba}^{-1} = a \cdot ab = a^2b$

② $\{e, ab, a^3b, a^2\} = N_2$

$a(ab)a^{-1} = a^3b$

index 2

③ $\{e, a, a^2, a^3\} = N_3$

④ $\{e, a^2\} = N_4 = Z(G)$

$\Leftrightarrow a^2b = ba^2$

other subgroups.

$$\left. \begin{array}{l} \{e, b\} \\ \{e, ab\} \\ \{e, a^2b\} \\ \{e, a^3b\} \end{array} \right\} \cong \mathbb{Z}_2$$

not normal.

For ①. ②. ③. $|N_i| = 4$ $\quad |G/N| = 2 \quad G/N \cong \mathbb{Z}_2$

① $N_1 = \{e.\ b.\ a^2 b.\ a^2\}$

$N_1 \cong D_2 \cong V$ $\quad \cong \begin{cases} \mathbb{Z}_4 \\ V \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_2 \end{cases}$

$(A = a^2.\ \ B = b \quad \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle)$

$D_4/N_1 = \{N_1,\ aN_1\} \cong \mathbb{Z}_2 = \{\pm 1\}$

$N_1 \cdot N_1 = N_1 \qquad N_1 \longrightarrow 1$

$N_1 \cdot (aN_1) = aN_1 \qquad aN_1 \longrightarrow -1$

$(aN_1) \cdot (aN_1) = a^2 N_1 = N_1$

	N_1	aN_1
N_1	N_1	aN_1
aN_1	aN_1	N_1

② $N_2 = \{e,\ ab,\ a^2,\ a^3 b\} \cong D_2 \quad (A = a^2,\ B = a^3 b)$

③ $N_3 = \{e, a, a^2, a^3\} \cong \mathbb{Z}_4$

$D_4/N_3 = \{N_3,\ bN_3\} \cong \mathbb{Z}_2$

④ $N_4 = Z(D_4) = \{e.\ a^2\}$ $\qquad (aZ)\cdot(aZ) = a^2 Z = \{a^2, e\} = Z$

$D_4/Z(D_4) = \{Z(D_4),\ aZ(D_4).\ bZ(D_4),\ abZ(D_4)\}$

$\cong D_2$

D_4 is nonabelian. $\Rightarrow$ $D_4/Z(D_4)$ non cyclic

[HW]: $G/Z(G)$ cyclic $\Leftrightarrow$ G is abelian.

4. determinant of A in $GL(n, K)$

$$GL(n, K) \xrightarrow{\det} K$$
$$A \longmapsto \det(A)$$

$[\ \det(AB) = \det(A)\det(B)\ ]$

$\ker(\det) = SL(n, K)$

$\Rightarrow SL(n, K) \triangleleft GL(n, K)$

$(\ \det(gAg^{-1}) = \det(A)\ )$

① $GL(n, K)/SL(n, K) \cong K^*$ $\quad M \in GL$

$\det M = z = re^{i\theta}$

$M = (r^{\frac{1}{n}} e^{i\theta/n}) \cdot A \quad A \in SL$

② $U(n)/SU(n) \cong U(1)$ $\quad U(n)$: $AA^* = 1$

$|\det A| = 1$

SU: $\det = 1$

③ $O(n)/SO(n) = \{SO(n), PSO(n)\} \cong \mathbb{Z}_2$

$(\det P = -1)$

5. Euclidean group E^3

$g = \{R_\alpha | \vec{\tau}\}$ $\quad g\cdot\vec{r} = R_\alpha \cdot \vec{r} + \vec{\tau}$

$$\{e|\vec{0}\} = \underbrace{\{R_\alpha|\tau\}}_{g}\underbrace{\{R_\beta|\tau'\}}_{g^{-1}} = \{\underbrace{R_\alpha R_\beta}_{e} | \underbrace{R_\alpha\tau' + \tau}_{\vec{0}}\}$$

$\Rightarrow \quad g^{-1} = \{R_\alpha^{-1} | -R_\alpha^{-1}\tau\}$

Consider the translation subgroup $T := \langle \vec{t}_1, \vec{t}_2, \vec{t}_3 \rangle$

($\vec{t}_i$ primitive lattice vectors) $\{e|t\} \in T$

$\{R_\alpha|\tau\}\{e|t\}\{R_\alpha^{-1} | -R_\alpha^{-1}\tau\}$

$= \{R_\alpha|\tau\}\{R_\alpha^{-1} | -R_\alpha^{-1}\tau + t\}$

$= \{e | R_\alpha(-R_\alpha^{-1}\tau + t) + \tau\}$

$= \{e | R_\alpha t\} \in T^3$

$\Rightarrow \quad g T^3 g^{-1} = T^3 \quad \forall g \in G.$

$\Rightarrow T^3 \triangleleft E^3$

6 $\{1\} \triangleleft G$, $\quad G \triangleleft G \quad$ trivial normal subgroups

(Def) A group with no nontrivial normal subgroups is called a simple group.

① $\mathbb{Z}_p \cong \mu_p$ with p prime $\quad H \subset \mathbb{Z}_p \quad |H| = 1$ or p

$H = \{1\}$ or $\mathbb{Z}_p$

② Alternating groups A_n

$A_3 \cong \mathbb{Z}_3$ A_3 is simple

$D_2 \cong V \triangleleft A_4$ A_4 is not simple

$A_{n\geq 5}$ are simple

6.4 Quotient groups and short exact sequences (Moore §7.4)

Theorem (1st isomorphism theorem) Rotman

$\mu : G \to G'$ homomorphism. with kernel K

$\Rightarrow$ $K \trianglelefteq G$, and $G/K \overset{\mu}{\cong} \mathrm{im}(\mu)$

Proof. $\varphi : G/K \longrightarrow \mathrm{im}\,\mu$

$gK \longmapsto \mu(g)$

① φ is well-defined. ($g_1K = g_2K \Rightarrow \mu(g_1) = \mu(g_2)$) $\varphi(g_1K) = \varphi(g_2K)$

$g_1K = g_2K \Rightarrow \exists k \in K \;\; g_1 = g_2 k$

$\Rightarrow g_2^{-1} g_1 = k \in K$

$\Rightarrow \mu(g_2^{-1} g_1) = \mu(g_2)^{-1}\mu(g_1) = 1_{G'}$

$\Rightarrow \mu(g_1) = \mu(g_2)$

② φ is a homomorphism.

$\underline{\varphi(g_1K \cdot g_2K)} = \varphi(g_1g_2K) = \mu(g_1g_2)$

$= \mu(g_1)\mu(g_2) = \underline{\varphi(g_1K)\varphi(g_2K)}$

③ a. $\mathrm{im}\,\varphi = \mathrm{im}\,\mu$ surjective

b. $\varphi(g_1K) = \varphi(g_2K) \overset{!}{\Leftrightarrow} \mu(g_1) = \mu(g_2)$ injective

RHS $\Leftrightarrow \mu(g_1 g_2^{-1}) = 1_{G'}$

$\Rightarrow g_1 g_2^{-1} \in K$

a+b : φ is an isomorphism. $\Rightarrow g_1K = g_2K$

Summary:

$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[dr, "\nu"', "\text{surj.}"] & G' \\ & G/K \arrow[u, "\varphi"', "\text{inj.}"] \end{tikzcd}$$

$\mu = \varphi \circ \nu$ commutative

$\nu : g \longmapsto gK$

Now we introduce a sequence of homomorphisms

$$\cdots\ G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} \cdots$$

The sequence is exact at G_i if

$$\operatorname{im} f_{i-1} = \ker f_i$$

A short exact sequence (SES) 正合序列 is of the form

$$1 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} 1$$
$$0 \qquad\qquad\qquad\qquad\qquad 0$$

① 1 represents trivial group. $\{1\}$

0 : abelian groups "+" as group multiplication

② $1 \longrightarrow G_1$: inclusion map.

$G_3 \longrightarrow 1$: trivial homomorphism

} unique

Exactness at G_i :

1 G_1 : $\ker f_1 = \{1_{G_1}\} \Rightarrow f_1$ is injective

2. G_2 : $\ker f_2 = \operatorname{im} f_1$

3. G_3: $\ker f_3 \equiv G_3 = \operatorname{im} f_2 \Rightarrow f_2$ is surjective