Recap:

1. even / odd permutations $\Leftrightarrow$ transpositions (ij)

equiv. $\phi = \sigma_1 \cdots \sigma_t \in S_n$ a cycle decomposition

$\operatorname{sgn} \phi = (-1)^{n-t}$

$\phi = (234)(78) = (1)(234)(5)(6)(78) = (-1)^{8-5} = -1$

$\operatorname{sgn}(\phi_1)\operatorname{sgn}(\phi_2) = \operatorname{sgn}(\phi_1\phi_2) \Rightarrow \operatorname{sgn}(\phi) = (1)\cdot(-1) = -1$

$\operatorname{sgn}: S_n \longrightarrow \mathbb{Z}_2$

$\phi \longmapsto \operatorname{sgn}(\phi) \in \{\pm 1\}$

2. $A_n \subset S_n$. $|A_n| = \frac{n!}{2}$ $\operatorname{sgn} = 1$

3 B_n: $\tilde{\sigma}_i = (i, i+1)$

same as $\sigma_i = (i \; i+1)$:

① $\tilde{\sigma}_i \tilde{\sigma}_j = \hat{\sigma}_j \hat{\sigma}_i$ $|i-j| \geq 2$

② $\hat{\sigma}_i \tilde{\sigma}_{i+1} \hat{\sigma}_i = \hat{\sigma}_{i+1} \hat{\sigma}_i \tilde{\sigma}_{i+1}$

difference: $\sigma_i^2 = 1$

$\hat{\sigma}_i^2 \neq 1$

hom. $B_n \longrightarrow S_n$ kernel?

$\hat{\sigma}_i \longmapsto \sigma_i$

4. cosets. $H \subset G$ a subgroup. $g \in G$.

$gH = \{gh \mid h \in H\}$ left coset

as group action: H right acts on G.

why not G left acts on H?

recall def of group action:

hom. $\Phi: G \to \underline{\underline{S_X}}$

cosets $\Leftrightarrow$ orbits. same or disjoint

if $g \in g_1H \cap g_2H$ then $g_1H = g_2H$

5. Lagrange: $|H| \mid |G|$

$[G:H]$ index of H in G.

set of cosets: G/H. homogeneous space.

"stronger" version: Sylow's theorem.

$p^k \mid |G|$. then G has

subgroups of order p^k

S_3: $= 6 = 2 \times 3$ $\quad \mathbb{Z}_2$, A_3 $\quad$ <u>trivial</u>: $\{1\}$

$Q = \{\pm 1, \pm i, \pm j, \pm k\}$ $\quad |Q| = 8 = 2^3$

2: $\{\pm 1\}$ $\quad$ 4: $\{\pm 1, \pm i\}$ $\quad$ 8: Q.

6.2 Conjugacy

Definition (a) a group element h is conjugate to h'

$\exists g \in G$. s.t. $h' = g h g^{-1}$

(b) conjugacy defines an equivalence relation.

The equivalence class is called the

conjugacy class (of h)

$C(h) := \{ g h g^{-1} : \forall g \in G \} \left(=: h^G\right)$

(c) $H \subset G$ is a subgroup. its conjugate

$H^g := g H g^{-1} = \{ g h g^{-1} : h \in H \}$ is also a subgroup

① $e \in H^g$ $\quad g e g^{-1} = e$

② $(g h_1 g^{-1})(g h_2 g^{-1}) = g(h_1 h_2) g^{-1} \in H^g$

③ $I(g h_1 g^{-1}) = g h_1^{-1} g^{-1} \in H^g$

Example.

1. Permutations ϕ_1, ϕ_2 are conjugate if they have the same cycle decomposition structure.

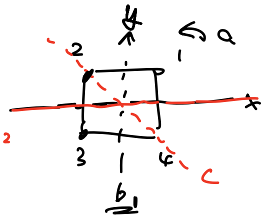
$$(a_1 a_2)(a_3 a_4 a_5) \sim (b_1 b_2)(b_3 b_4 b_5)$$

$$\Rightarrow \tau (a_1 a_2 \cdots a_n) \tau^{-1} = (\tau(a_1), \tau(a_2), \cdots \tau(a_n))$$

$$\tau (a_1 a_2)(a_3 a_4 a_5) \tau^{-1} = (b_1 b_2)(b_3 b_4 b_5)$$

$$\Leftrightarrow \boxed{\tau(a_i) = b_i}$$

2. $D_4 := \langle a, b : a^4 = b^2 = 1, (ab)^2 = 1 \rangle$



$a = (1234)$

$b_1 = (12)(34)$

$c = ab = (1234)(12)(34) = (13)(2)(4) = (13)$

$$c b_1 c^{-1} = (13)(12)(34)(13) = (14)(23) = b_2 \qquad b_1 \sim b_2$$

$$D_4 = \{1\} \cup \{a^2\} \cup \{a, a^3\} \cup \{b, a^2 b\} \cup \{ab, a^3 b\}$$

$$= \{()\} \cup \{(13)(24)\} \cup \{(1234), (1432)\}$$

$$\cup \{(12)(34), (14)(23)\} \cup \{(13), (24)\}$$

$$\left(\begin{array}{l} \tau (13)(24) \tau^{-1} = (12)(34) \\ \tau(3) = 2 \quad \tau(2) = 3 \qquad \tau = (23) \end{array} \right)$$

3. in $GL(n.K)$:

$$G = U(n) := \{ A \in M_n(\mathbb{C}) \mid AA^\dagger = 1_n \}$$

conj. are similarity transformations. How to label conj. classes?

Spectral theorem ensures $u \in U(n)$ can be diagonalized as. $\exists g \in U(n)$

↑ finite dim.

$$g u g^{-1} = \text{diag}(\underline{z_1 \cdots z_n}) \quad (|z_i| = 1)$$

$U(1)^n$

↪ conjugacy classes labeled by $(z_1 \cdots z_n)$?

permutation $A(\phi)\, \text{diag}(\underline{z_1 \cdots z_n})\, A(\phi)^{-1}$

$$= \text{diag}(\underline{z_{\phi(1)}, z_{\phi(2)} \cdots z_{\phi(n)}})$$

$$[A(\phi) g] u [A(\phi) g]^{-1} = \text{diag}\{ z_{\phi(i)} \}$$

$\Rightarrow$ $U(1)^n / S_n$ labels conj. class.

4. a general element of $GL(n.\mathbb{C})$ is not diagonalizable. Define the characteristic polynomial $(A \in GL(n))$

$$P_A(x) = \det(x1 - A)$$

$$P_{gAg^{-1}}(x) = \det(x1 - gAg^{-1})$$

$$= \det(g(x1 - A)g^{-1})$$

$$= \det(x1 - A) = P_A(x)$$

<u>Definition</u> A <u>class function</u> on a group is a function f on G, s.t.

$$f(g g_0 g^{-1}) = f(g_0) \qquad \forall g, g_0 \in G.$$

For a matrix representation, define the <u>character</u> of the representation

$$\chi_T(g) := \operatorname{Tr} T(g)$$

It is a class function.

<u>Definition</u>. Two homomorphisms $\varphi_i : G_1 \to G_2$ are conjugate if $\exists g_2 \in G_2$, s.t.

$$\varphi_2(g_1) = g_2\, \varphi_1(g_1)\, g_2^{-1} \qquad (\forall g_1 \in G_1)$$

in terms of representations $(T: G \to GL(V))$

$$\begin{tikzcd} V_1 \arrow[r, "S"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "S"] & V_2 \end{tikzcd}$$

(equivariant map / morphism of G-space)

$$T_2(g) S = S T_1(g) \qquad (\dim V_1 = \dim V_2)$$

$$T_2(g) = S T_1(g) S^{-1} \Leftarrow \text{equivalent representation}$$

5. conjugacy classes in S_n. (§ 7.5 of Moore)

Permutations with same structure of cycle decomposition are conjugate.

The conjugacy classes are labeled by the cycle decomposition of their elements. $C(\vec{l})$

$\vec{l} = (l_1, l_2, \cdots, l_n)$ where l_r is the number of r-cycles.

$$n = \sum_{j=1}^{n} j \cdot l_j$$

$\phi = (12)(34)(678)(11,12) \in S_{12}$

$= (12)(34)(5)(678)(9)(10)(11,12)$

$\vec{l} = \begin{array}{cccc} l_1 & l_2 & l_3 & l_{\geq 4} \\ 3, & 3 & 1 & 0 \end{array}$ $\quad \vec{l} = (3, 3, 1, 0, \cdots 0)$

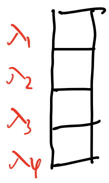
$\Rightarrow$ The number of conjugacy classes of S_n is given by the partition function of n, $\underline{p(n)}$. namely the number of distinct partitions of n into sum of nonnegative integers.

Example S_4

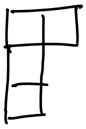
partition	cycle decomp.	tyical g	$\lvert C(g) \rvert$	order of g
$4 = 1+1+1+1$	$(1)^4$	1	1	1
$4 = 1+1+2$	$(1)^2(2)$	(ab)	$\binom{4}{2} = 6$	2
$4 = 1+3$	$(1)(3)$	(abc)	$2\binom{4}{3} = 8$	3
$4 = 2+2$	$(2)^2$	$(ab)(cd)$	$\frac{1}{2}\binom{4}{2} = 3$	2
$4 = 4$	(4)	$(abcd)$	6	4

$|S_4| = 24 = 1+6+8+3+6$ $\qquad p(4) = 5$

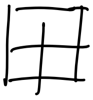
Young diagram : $\quad n = \sum_{i=1}^{k} \lambda_i \qquad \lambda_i \geq \lambda_{i+1} \geq 0$



$(1)^4$



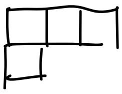
$(1)^2(2)$



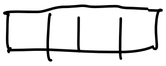
$(2)^2$

$\downarrow$ λ_i : number of boxes in i-th row

$\lambda_i \geq \lambda_{i+1}$.



$(1)(3)$



(4)

Define a partition : $\lambda = \{\lambda_1, \lambda_2 \cdots \}$ $\quad n \stackrel{k}{\underset{i=1}{=}} \Sigma \lambda_i$

multiplicity : $m_i(\lambda) = |\{ j \mid \lambda_j = i \}|$

label conj. class $\quad C(\lambda) := (1)^{m_1} (2)^{m_2} \cdots$