Recap:

1. Group representations

$$\varphi : G \longrightarrow GL(V) \xrightarrow{b} GL(n, k)$$

V: carrier space. $\qquad n = \dim_k V$.

$$g \longmapsto \tau_g \longmapsto \Gamma(g)$$

equivalent rep: $\exists S$. s.t. $\forall g \in G$. $\Gamma'(g) = S\Gamma(g)S^{-1}$

2. group actions on sets X

hom. Φ : $G \longrightarrow S_X := \{ X \xrightarrow{f} X : f$: permutations (1-1 & onto)

$g \longmapsto \phi(g, \cdot)$ a permutation.

$$\phi(g_2, \phi(g_1, x)) \equiv g_2 \cdot (g_1 \cdot x)$$
$$= (g_2 \cdot g_1) \cdot x \equiv \phi(g_2 g_1, x)$$

X: G-set

Examples: 1. $X = G$. then

$g \circ x \equiv g \cdot x$ group multi.

$g \circ x \equiv g x g^{-1}$ conj.

2. $GL(n, k)$ on k^n. $\longrightarrow$ SG. on $\vec{r} \in \mathbb{R}^3$

<u>orbits</u> : $\mathcal{O}_G(x) = \{ g \cdot x \mid \forall g \in G \}$

$\mathcal{O}_G(x_1) \cap \mathcal{O}_G(x_2) = \emptyset$ or $\mathcal{O}_G(x_1) = \mathcal{O}_G(x_2)$

$\Rightarrow$ equivalence classes

X/G. partitions G.

3. induced action on function space:

$f \in F[X \to Y]$

$(g \cdot f)(x) = f(g^{-1} \circ x)$

$\Rightarrow (g \circ f)(g \circ x) = f(x)$ ✓

4.3 equivariant maps

Definition Let X, X' be two G-spaces

A equivariant map, $f: X \to X'$

satisfies

$$f(g\cdot x) = g\cdot f(x) \qquad \forall x\in X \ \forall g\in G.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

$$f(\phi(g, x)) = \Phi'(g, f(x))$$

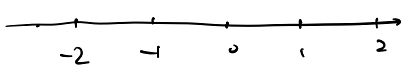
f is also called a morphism of G-spaces. 态射：保持数学结构不变

Examples.

$G = \mathbb{Z}$ acts on $\mathbb{R}$

$n: x \mapsto x+n$

orbits?



$\mathbb{R}/\mathbb{Z} = [0,1) \sim S^1$

- equivariant map?

$f : \mathbb{R} \to \mathbb{R}$

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ \phi_n \downarrow & & \downarrow \phi_n \\ \mathbb{R} & \xrightarrow{f} & \mathbb{R} \end{array}$$

$$f(x) + n_1 = f(x+n_1)$$

$$f(x) + n_2 = f(x+n_2)$$

$$f(\underline{x}+n_1) - f(\underline{x}+n_2) = \underline{n_1 - n_2} \qquad \forall x, n_i$$

$$f(x) = x + \alpha$$

5. The symmetric group

(Moore Sec. 6)

Rotman. Intro. to the theory of groups

陶 / 3

Recall that

Given a set X, the set of all permutations

$$S_X := \{ X \xrightarrow{f} X : f : 1\text{-}1 \ \& \text{ onto (invertible)} \}$$

For $n \in \mathbb{N}^+$ denote the symmetric group on n elements, S_n, which is the set of all permutations of the set $X = \{1, 2, \cdots, n\}$

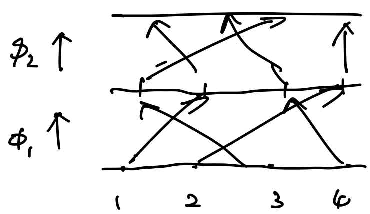
$(|S_n| = n!)$

A permutation can be written as (two-line notation)

$$\phi = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix} \quad \text{with } p_i = \phi(i)$$

$$\phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$\phi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$



$$\phi_2 \cdot \phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

we use the "right" convention.

$$\phi_2 \cdot \phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \equiv (2\ 4)$$ 循环表示

Canonical permutation rep. of S_n

"regular rep. of S_n" by Zee book. 正则表示

Consider S_n and an n-dim. carrier space ($\mathbb{R}^n$, $\mathbb{C}^n$ etc.)

with an ordered basis $\vec{e}_i = \{0, 0, \cdots 0, \underset{i\text{-th}}{\underline{1}}, 0, \cdots 0\}$

$\phi \in S_n$: $T(\phi): \vec{e}_i \longrightarrow \vec{e}_{\phi(i)}$

$$T(\phi)\vec{e}_i = \sum_{j=1}^{n} A(\phi)_{ji}\, \vec{e}_j \qquad A \in GL(n, K)$$

$$A_{ji}(\phi) = \vec{e}_j^{\,T} \cdot \vec{e}_{\phi(i)} = \delta_{j, \phi(i)}$$

$\phi = (1234) \in S_4$ $\quad A(\phi) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$\phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ $\quad A(\phi_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$\phi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ $\quad A(\phi_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

5.1. Cycles & transpositions

Definition. Let $i_1, \cdots i_r$ be distinct integers between 1 and n.

If $\phi \in S_n$ fixes the remaining integers and if

$$\phi(i_1) = i_2 , \ \phi(i_2) = i_3 , \ \cdots \ \phi(i_r) = i_1$$

then ϕ is an r-cycle (cycle of length r)

$$(i_1 \, i_2 \, i_3 \cdots i_r)$$

A 2-cycle is called a transposition.

Remarks:

1. cycles are the same up to cyclic ordering

$$(2\,3\,4) = (4\,2\,3) = (3\,4\,2)$$

2. disjoint cycles commute

$$(2\,3\,4)(5\,6) = (5\,6)(2\,3\,4)$$

$$\underset{(2\,3\,1)}{(1\,2)(2\,3)} \neq \underset{(1\,3\,2)}{(2\,3)(1\,2)}$$

3. inverse of a permutation

$$[(1\,2)(3\,4\,5)]^{-1} = (1\,2)(5\,4\,3) = (1\,2)(3\,5\,4)$$

$$[(1\,2)(2\,3)]^{-1} = (2\,3)(1\,2) = (1\,3\,2) = (2\,3\,1)^{-1}$$

Theorem: Every permutation $\phi \in S_n$ is either a cycle or can be factorized into disjoint cycles.

(Proof by induction)

(Def) complete factorization: is a product of disjoint cycles which contains one 1-cycle for each fixed x.

$$\underline{(1)(234)} \quad \Big(\neq (1)(1)(234)\Big)$$

Complete factorization of a permutation ϕ is unique (up to ordering). which we call the cycle decomposition of ϕ.

5.2. Cayley's theorem

Theorem (Cayley, 1878)

Every group G is isomorphic to a subgroup of S_G ("can be embeded in S_G")
In particular, if $|G| = n$, then G is isomorphic to a subgroup of S_n.

Proof: recall group action. let $X = G$. define left-mult.

$$\forall h. \quad L(h): G \longrightarrow G$$
$$g \longmapsto h \cdot g$$

$L(h) \in S_G$. as it is one-one and onto

and naturally $L(h_1) \cdot L(h_2) = L(h_1 h_2)$

So the map $L: h \longmapsto L(h)$ is a homomorphism.

L is one-one. thus $G \cong L(G) \subset S_G$

($S_G \cong S_N$ with an ordered set.

$\{1, \omega, \omega^2, \cdots \omega^{N-1}\} =: \mu_N \qquad S_{\mu_N} \cong S_N$
"natural ordering"

D_n, $SU(n)$. has no natural order)

Example. $\mathbb{Z}_n \cong \langle \underbrace{(1\,2\,\cdots\,n)}_{\langle w \rangle} \rangle \cong \mu_n$

$n=3$

$\langle (123) \rangle = \{1, (123), (132)\} = A_3 \subset S_3$ (later)

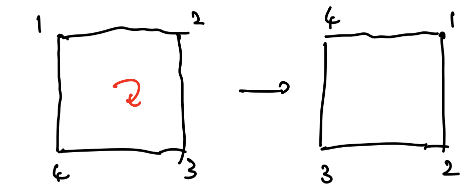
$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$\mu_3 \quad \{1, \quad w \quad w^2\}$

Example. $D_4 = \langle A B \mid A^4 = B^2 = (AB)^2 = 1 \rangle$

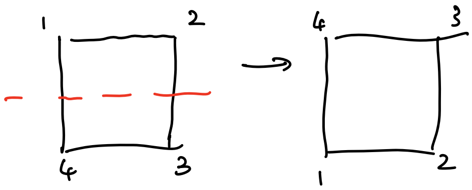
$|D_4| = 8 \cong$ a subgroup of $\underline{S_8}$

A :



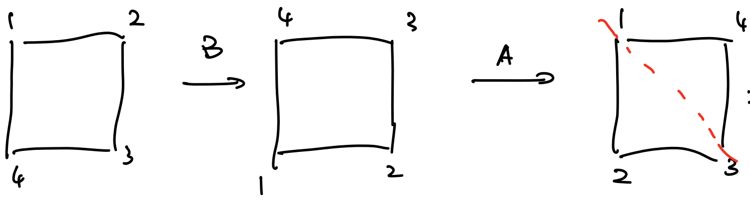
$:= (1\,2\,3\,4)$ subgroup of $\underline{\underline{S_4}}$

B :



$:= (14)(23)$

AB :



$:= (24)$

$(AB)^2 = 1$

How to find the isomorphism ?

$\to$ use multiplication table (Cayley table)

Klein's 4-group. $V = \langle a b \mid a^2 = b^2 = (ab)^2 = e \rangle$

$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$e = (0, 0)$

$a = (1, 0)$ $|V| = 4$

$b = (0, 1)$

$c = (1, 1)$

$\phi : V \longrightarrow im(V) \subset S_4$

$a \longmapsto \phi(a)$

		e	a	b	c
→ 1	e	e	a	b	c
2	a	a 2	e 1	c 4	b 3
3	b	b 3	c 4	e 1	a 2
4	c	c	b	a	e

$\phi(e) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$= 1$

$\phi(a) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$= (12)(34)$

$\phi(b) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

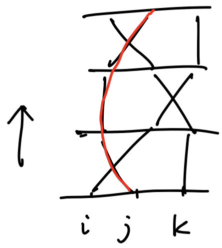
$= (13)(24)$

$\phi(c) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$

I.3. Transpositions / 2-cycles

i, j, k are distinct.

① $(i\,j)(j\,k)(i\,j) = (i\,k) = (j\,k)(i\,j)(j\,k)$



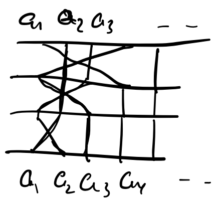
② $(i\,j)^2 = 1 \qquad (i\,j) = (i\,j)^{-1}$

③ $(i\,j)(k\,l) = (k\,l)\cdot(i\,j) \qquad \{i, j\} \cap \{k, l\} = \phi$

Theorem. Every permutation $\phi \in S_n$ is a product of transpositions. (c.f. cycle decomp.)

Proof. $\phi \in S_n$ has a cycle decomposition.

For each cycle,



$$(a_1\, a_2 \cdots a_r) = (a_1\, a_r)(a_1\, a_{r-1}) \cdots (a_1\, a_2)$$

any permutation can be generated by transpositions

Remarks:

1. There are other ways to generate S_n

① $\sigma_i = (i \;\; i+1) \quad (1 \le i \le n-1)$

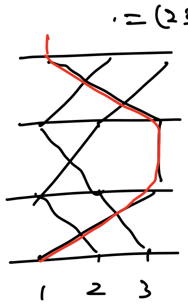
"elementary generators"

$(ij) = (i, i+1)(i+1, j)(i, i+1) \quad (i < j)$

② generated by (12) & $(12 \cdots n)$

$(23) = (12 \cdots n)(12)(1 \cdots n)^{-1}$

$:= (23)$



Remark: transposition decomposition is not unique

$(123) = \underbrace{(13)(12)}_{2} = \underbrace{(23)(13)}_{2}$

$= \underbrace{(13)(42)(12)(14)}_{4}$

$= \underbrace{(13)(42)(12)(14)(23)(23)}_{6} \cdots$

always even number of transpositions