Recap:

1. $\varphi: G \to H$

$\ker\varphi\,(\equiv K) := \{g \in G \mid \varphi(g) = 1_H\} \subset G$ subgroups

$\mathrm{im}\,\varphi := \varphi(G) \subset H$

iso. $\left.\begin{array}{l}\ker\varphi = \{1_G\} \\ \mathrm{im}\,\varphi = H\end{array}\right\} G \cong H$

2. $SU(2) \longrightarrow SO(3)$

$R: u \mapsto R(u)$

s.t. $u(\vec{x}\cdot\vec{\sigma})u^{-1} = (R(u)\cdot\vec{x})\cdot\vec{\sigma}$

$u(x^i\sigma_i)u^{-1} = (R(u)^j{}_i x^i)\sigma_j \quad \forall \vec{x} \in \mathbb{R}^3$

$u\sigma_i u^{-1} = R(u)^j{}_i\sigma_j$

▷ recall : ($u_\theta^\dagger S_\alpha u_\theta = R_{\theta\,\alpha}{}^\beta S_\beta$ spin rotation)

$R(u) = R(-u)$ 2 to 1 mapping

$SU(2)$ is a double cover of $SO(3)$

$\ker R = \{\pm 1\} \cong \mathbb{Z}_2$

$u(\hat{n},\theta) = \exp(-\frac{i}{2}\hat{n}\cdot\vec{\sigma}) \in SU(2) \quad u(\hat{n},2\pi) = -1 \;.\; u(\hat{n},4\pi) = 1$

$SU(2) \cong S^3$

$SO(3) \cong \mathbb{RP}^3$: S^3 with antipodes identified.

Example. $GL(V)$ and $GL(n, K)$

Let $GL(V): V \to V$ be the group of invertible linear transformations with a finite dimensional vector space V.

Given an ordered basis $b = \{\hat{e}_1, \dots \hat{e}_n\}$

Define a homomorphism:

$$\varphi_b : GL(V) \longrightarrow GL(n, K)$$

$$\tau \longmapsto T_b(\tau)$$

$$\text{s.t.} \quad \underline{\tau(\hat{e}_i) = \sum_j \hat{e}_j \cdot T_b(\tau)_{ji}} \quad @$$

$\forall \vec{v} \in V$. $\vec{v} = \sum_{i=1}^{n} v_i \hat{e}_i \qquad (v_i \in K)$

$$\tau \vec{v} = \sum_{i=1}^{n} v_i (\tau \hat{e}_i) = \sum_{ij} \hat{e}_j \cdot T_b(\tau)_{ji} v_i$$

$$\Rightarrow \tau_1(\tau_2 \vec{v}) = \sum_{ij} (\tau_1 \hat{e}_j) T_b(\tau_2)_{ji} v_i$$

$$= \sum_{ijk} \hat{e}_k \cdot T_b(\tau_1)_{kj} T_b(\tau_2)_{ji} v_i$$

$$= \sum_{ik} \hat{e}_k [T_b(\tau_1) T_b(\tau_2)]_{ki} v_i$$

$$\equiv (\tau_1 \tau_2) \vec{v}$$

$$= \sum_{ik} \hat{e}_k [T_b(\tau_1 \tau_2)]_{ki} v_i$$

$$\Rightarrow T_b(\tau_1 \tau_2) = T_b(\tau_1) T_b(\tau_2)$$

$\begin{cases} \text{surjective } \checkmark \\ \text{injective ? } \quad \tau(\hat{e}_i) = e_i \Leftrightarrow \tau = id \quad \checkmark \end{cases}$

$$\Updownarrow$$

$$T_L(\tau) = \mathbb{1}_n$$

<u>isomorphism</u> $\quad \underline{GL(V)} \stackrel{u}{\cong} \underline{GL(n, K)}$

<u>Definition</u>.

① Let G be a group. then a finite dimensional <u>representation</u> of G is a finite dimensional vector space V with a group homomorphism

$\varphi : G \longrightarrow GL(V)$

V: carrier space

② A matrix representation of G is a homomorphism

$$\varphi : G \longrightarrow GL(n, K) \qquad (K = \mathbb{R}, \mathbb{C})$$

$$g \longmapsto \underline{P}(g)$$

$$\forall g_1, g_2 \in G : \quad P(g_1 g_2) = P(g_1) P(g_2)$$

① + an ordered basis $\longrightarrow$ ② $\quad (GL(V) \stackrel{u}{\cong} GL(n, K))$

Matrix rep. is basis dependent

$$\hat{e}_i = \sum_{j=1}^{n} S_{ji}\, \hat{e}'_j$$

$$\Gamma'(g) = S\,\Gamma(g)\,S^{-1}$$

<u>Definition</u> (equivalent representation) Γ, Γ' are n-dim reps of G

Γ, Γ' are equivalent ($\Gamma \cong \Gamma'$) if $\exists S \in GL(n,k)$ s.t. $\forall g \in G$ $\quad \Gamma'(g) = S\,\Gamma(g)\,S^{-1}$

<u>Example</u> $\mathbb{Z}, \mathbb{R}, \mathbb{C} \ni a$

$$\Gamma(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

$$\Gamma(a)\Gamma(b) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix}$$

<u>Example</u>. $S_2 = \{e, \sigma\} \qquad \sigma^2 = e$

$$S_2 \cong \mu_2 \cong \mathbb{Z}_2$$

$$\Gamma(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Gamma(\sigma^2) = \Gamma(\sigma)\cdot\Gamma(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example $\mu_3 = \langle \omega \mid \omega^3 = 1 \rangle$

$\Gamma(e) = \mathbb{1}_3$

$$\Gamma(\omega) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Gamma(\omega^2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Example $D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$ ✓

$|D_4| = 8$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$C = AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Gamma(D_4) = \{ \pm \mathbb{1}, \pm A, \pm B, \pm C \}$$

isomorphism : "faithful representation"

not faithful. $\Gamma(A) = \Gamma(B) = \mathbb{1}$

4. Group actions on sets (Moore Sec. 5)

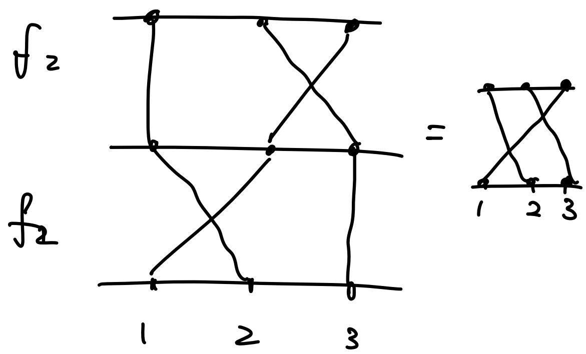
Definition: Given a set X, the set of permutations 排列/置换

$$S_X := \{ X \xrightarrow{f} X : f : 1-1 \ \& \ \text{onto (invertible)} \}$$

is a group under composition.

$$m(f_1, f_2) := f_1 \circ f_2$$

$$\begin{tikzcd} X \arrow[r, shift left, "f_2"] \arrow[rr, bend right, "f_1 \circ f_2"'] & X \arrow[l, shift left, "f_2"] \arrow[r, shift left, "f_1"] & X \arrow[l, shift left, "f_1^{-1}"] \end{tikzcd}$$



Definition. A (left) group action Φ of G is a homomorphism

$$\Phi : G \longrightarrow S_X$$

$$g \longmapsto \underline{\phi(g, \cdot)}$$

$\phi(g, \cdot) : X \to X$

$x \mapsto \phi(g, x)$

1. $\phi : G \times X \longrightarrow X \qquad \phi(g, x) \in X \quad (\forall x \in X)$

$$\phi(g_1, \phi(g_2, x)) = \phi(\underline{g_1 g_2}, x)$$

$$\phi(1_G, x) = x \quad (\forall x \in X)$$

$$\phi(g, \phi(g^{-1}, x)) = \phi(g g^{-1}, x) = \phi(1_G, x) = x$$

simplified notation: $g \cdot x := \phi(g, x)$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad (\forall x \in X)$$

Definition : If a set X has a group action by G we say that X is a **G-set**.

Example 1. $X = G$.

① group action by **multiplication**

$x \in X = G$

$g_1 \cdot (g_2 x) = g_1 g_2 x = (g_1 g_2) x$

② group action by **conjugation**

$g \cdot x := g x g^{-1} \in G = X$

a. $g_1 \cdot (g_2 \cdot x) = g_1 (g_2 x g_2^{-1}) = g_1 g_2 x g_2^{-1} g_1^{-1}$

$= (g_1 g_2) \cdot x$

b. $e \cdot x = e x e^{-1} = x$

Abelian group. $g \cdot x = g x g^{-1} = x \quad (\forall g \in G)$

2. $GL(n, k)$ acts on k^n.

$A \cdot \vec{v} = \sum_j A_{ij} v_j$

$e = 1_n$

a rep. of G $\Rightarrow$ group action on carrier space V.

3. Space group action on $\mathbb{E}^3$ ($\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ + Origin)
$\Rightarrow \mathbb{R}^3$

$\{g|\tau\}$ $\quad g \in O(3)$

$\tau \in T$ translation

$$\{R_g|\vec{\tau}\}\cdot\vec{r} := R_g\vec{r} + \vec{\tau} \qquad R_g \in O(3)$$

$$\underline{\{R_1|\vec{\tau}_1\}\{R_2|\vec{\tau}_2\}\cdot\vec{r}} = \{R_1|\vec{\tau}_1\}(R_2\vec{r} + \vec{\tau}_2)$$

$$= R_1(R_2\vec{r} + \vec{\tau}_2) + \vec{\tau}_1$$

$$= \underline{\{R_1R_2 | R_1\vec{\tau}_2 + \vec{\tau}_1\}\cdot\vec{r}}$$

matrix rep.

$$\{g|\vec{\tau}\} = \begin{array}{c} \left(\begin{array}{c|c} 1 & 0 \\ \hline \vec{\tau} & R_g \end{array}\right) \end{array} \begin{array}{c} 1 \\ 3 \end{array} \qquad \vec{r} \to \begin{pmatrix} 1 \\ \vec{r} \end{pmatrix}$$

$$\{R_1|\vec{\tau}_1\}\{R_2|\vec{\tau}_2\} = \begin{pmatrix} 1 & 0 \\ \tau_1 & R_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \tau_2 & R_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ R_1\tau_2 + \tau_1 & R_1R_2 \end{pmatrix}$$

$$= \{R_1R_2 | R_1\vec{\tau}_2 + \vec{\tau}_1\}$$

<u>Definition</u> (Orbits). Let X be a G-set

the orbit of G through a point $x \in X$ is

the set

$$O_G(x) := \{ g \cdot x \mid \forall g \in G \}$$

$$= \{ y \in X : \exists g, \text{ s.t. } y = g \cdot x \}$$

This defines an equivalence relation "$\sim$"

($x \sim x$; $x \sim y \Leftrightarrow y \sim x$; $x \sim y,\ y \sim z \Rightarrow x \sim z$)

$O_G(x)$ are equivalence classes ($[x]$) under group action.

Distinct orbits of G partition X :

① $\forall x (\in X) \in O_G(x)$

② If $O_G(x_1) \cap O_G(x_2) \ni x \Rightarrow O_G(x_1) = O_G(x_2)$

$$(x = g_1 \cdot x_1 = g_2 \cdot x_2 \qquad \underline{x_1} = g_1^{-1} \cdot g_2 \cdot \underline{x_2})$$

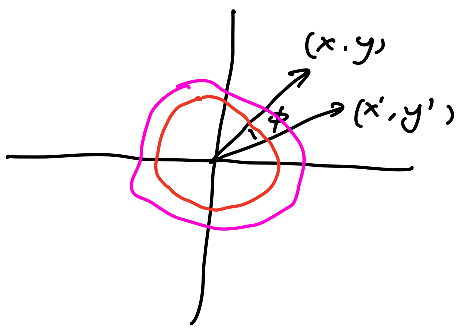
$\Rightarrow$ X is covered by disjoint orbits.

The set of orbits is denoted as X/G $\triangle$

Examples

1. $G = SO(2, \mathbb{R})$. on $\mathbb{R}^2$

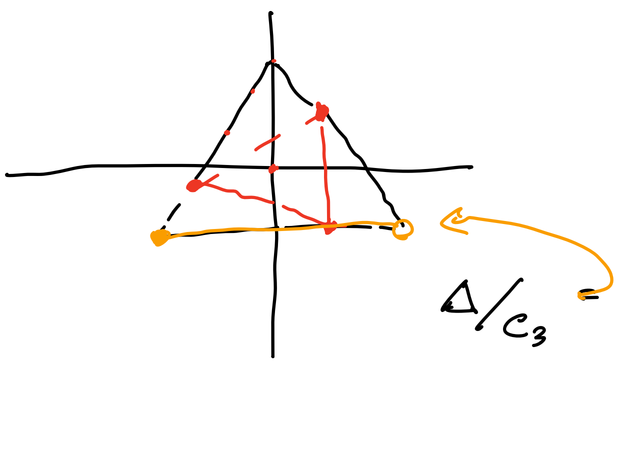
$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$



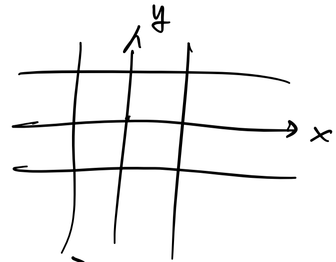
$$\mathbb{R}^2 / SO(2) = [0, +\infty)$$

2. $G = C_3 = \{ R(0), R(2\pi/3), R(4\pi/3) \} \cong \mathbb{Z}_3$
$\subset SO(2, \mathbb{R})$

X: eq lat. triangle



Example Space group acts on a 2D square lattice



$p4mm = \{ \{g|\vec{\tau}\} : g \in D_4 , \vec{\tau} \in a\hat{x} + b\hat{y} \ (a, b \in \mathbb{Z}) \}$

group presentations ?

$\langle R, m_x, T_x, T_y \rangle$

$\sim : \vec{r}_1 - \vec{r}_2 \in \{\vec{\tau}\}$

Consider positions:

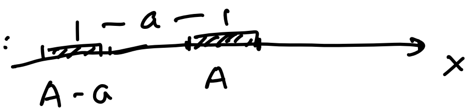
1a	$(0,0)$	$4mm$
1b	$(\frac{1}{2}, \frac{1}{2})$	$4mm$
2c	$(\frac{1}{2}, 0)\ (0, \frac{1}{2})$	m
4d	$(\pm x, 0)\ (0, \pm x)$	m
4e	$(\pm x, \frac{1}{2})\ (\frac{1}{2}, \pm x)$	m
4f	$(\pm x, \pm x)$	m
8g	(x, y)	1

different orbit types

Wyckoff positions

Consider wave function $\psi(\vec{r})$: $\psi : \mathbb{R}^3 \longrightarrow \mathbb{C}$

1D:



$$\psi' = T_a \psi$$

$$\int_A |\psi'(x)|^2 dx = \int_{A-a} |\psi(x')|^2 dx' \overset{x = x'+a}{=} \int_A |\psi(x-a)|^2 dx$$

$$\Rightarrow \psi'(x) = e^{i\theta_T} \psi(x-a)$$

$$= \rho(a) \psi(x-a)$$

4.2 Induced group actions on associated function spaces.

$\mathcal{F}[X \to Y]$ is the set of functions from set X to set Y. Let ϕ be the left group action

$$\phi : G \times X \to X.$$

Then there is also a group action $\tilde{\phi}$ on $\mathcal{F}$:

$$\tilde{\phi}(g, F)(x) := F(\phi(g^{-1}, x)) \qquad F \in \mathcal{F}.$$

$$\tilde{\phi}(g_1, \tilde{\phi}(g_2, F))(x) = \tilde{\phi}(g_2, F)(\phi(g_1^{-1}, x))$$

$$= F(\phi(g_2^{-1}, \phi(g_1^{-1}, x)))$$

$$= F(\phi((g_1 g_2)^{-1}, x))$$

$$= \tilde{\phi}(g_1 g_2, F)(x)$$

$$(g \cdot F)(x) := F(g^{-1} \cdot x)$$

4.3 equivariant maps

__Definition__ Let X, X' be two G-spaces

A __equivariant map__, $f : X \to X'$

satisfies

$$f(g \cdot x) = g \cdot f(x) \qquad \forall x \in X \ \forall g \in G.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

$$f(\phi(g.x)) = \Phi'(g.\ f(x))$$

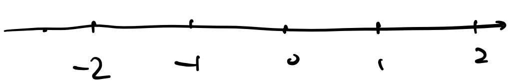
f is also called a morphism of G-spaces.

__Examples__:

$G = \mathbb{Z}$ acts on $\mathbb{R}$

$n : x \mapsto x + n$

orbits?



$\mathbb{R}/\mathbb{Z} = [0, 1) \sim S^1$

- equivariant map?

$f : \mathbb{R} \to \mathbb{R}$

$$\begin{array}{ccc} R & \xrightarrow{f} & R \\ {\scriptstyle \phi_n}\downarrow & & \downarrow{\scriptstyle \phi_n} \\ R & \xrightarrow{f} & R \end{array}$$

$$f(x)+n_1 = f(x+n_1)$$

$$f(x)+n_2 = f(x+n_2)$$

$$f(\underline{x}+n_1) - f\underline{(}x+n_2) = \underline{n_1-n_2} \qquad \forall x, n_i$$

$$f(x) = x+\alpha$$