Recap:

1. Group presentation. $G = < g_1 \cdots g_e \mid R_1 \cdots R_r >$

Dihedral group $D_n = < a, b \mid a^n = b^2 = (ab)^2 = 1 >$

Klein's 4-group $V = < a, b \mid a^2 = b^2 = (ab)^2 = 1 > \cong D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Pauli group: $P = < \sigma^1, \sigma^2, \sigma^3 >$

quaternion group: $Q_8 = < a b \mid a^4 = 1, a^2 = b^2, ba = a^{-1} b >$

$a = i, b = j$ ① $i^4 = 1$. ② $i^2 = j^2$ $(\neq 1)$

③ $ij = -ji$ what is "-1"?

$(ij)^2 = ijij = i(-ij)j = -i^2 j^2 = -1 = i^2$ ✓

$\underline{\underline{ji = -ij = i^{-1} j}}$ ✓

$ijij = i^2 \Rightarrow ij = i^2 j^{-1} i^{-1} = i^2 j (j^{-2}) i^{-1} = i^2 j i^{-3}$

$= i^2 (j^{-2}) j \cdot i^{-1} = j \cdot i^{-1}$

$\vdots$

all elements: $i^m j^n i^l j^k \cdots \Rightarrow i^m j^n$ $m, n \in [0, 3)$

$1, i, i^2, i^3, j, ij, i^2 j, i^3 j$ $i^2 := -1$. $ij := k$

$= 1, i, -1, -i, j, k, -j, -k$

What happens, if we drop one of the relations?

① removing $i^4 = 1$ $\Rightarrow$ infinite group. $i^m j$ $m \in \mathbb{Z}$

② removing $i^2 = j^2$. $\Rightarrow$ $i^m j^n$ $m \in [0.3)$

j not limited.

③ removing $(ij)^2 = i^2$. no relation between

ij multiplication.

$i^{n_1} j^{n_2} i^{n_3} j^{n_4} \cdots$

2. Homomorphism & isomorphism.

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "\varphi \times \varphi"'] & G \arrow[d, "\varphi"] \\ G' \times G' \arrow[r, "m'"] & G' \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "I"] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi"] \\ G \arrow[r, "I'"] & G \end{tikzcd}$$

$\underline{\underline{\varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2)}}$

$\Downarrow$

$\varphi(e)^2 = \varphi(e)$

$\underline{\underline{\varphi(e) = e'}}$

$\Downarrow$

$\varphi(g \cdot g^{-1}) = \varphi(g)\varphi(g^{-1}) = e'$

$\Downarrow$

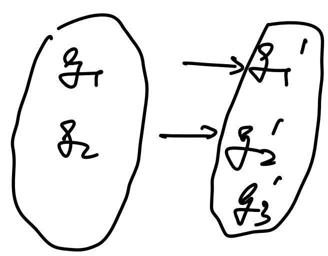
$\underline{\underline{\varphi(g)^{-1} = \varphi(g^{-1})}}$

① injective: $\forall g_1 . g_2 \in G$.

$\varphi(g_1) = \varphi(g_2)$ iff $g_1 = g_2$

$\Updownarrow$

$\varphi(g) = e'$ iff $g = e$



② surjective: $\forall g' \in G'$. $\exists g \in G$ s.t. $\varphi(g) = g'$

③ isomorphism: ① + ②

$$G \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} G' \quad \text{iso.}$$

$$G \cong G'$$

Definition (kernel & image)

φ hom. $\varphi : G \to H$

(a) kernel K

$$K := \ker \varphi := \{ g \in G : \varphi(g) = 1_H \}$$

(b) image

$$\operatorname{im} \varphi := \{ h \in H : \exists g \in G \text{ s.t. } \varphi(g) = h \}$$
$$= \varphi(G)$$

Remarks

(a) $\varphi(G) \subset H$. is a subgroup ? ✓

① $\varphi(1_G) = 1_H$

② $\forall.\ h_1 = \varphi(g_1)$. $h_2 = \varphi(g_2)$

$h_1 h_2 = \varphi(g_1)\varphi(g_2) = \varphi(\underline{g_1 g_2}) \in \varphi(G)$ ✓

③ $h_1 = \varphi(g_1)$ $\quad 1_H = \varphi(g_1 \cdot g_1^{-1}) = \underbrace{\varphi(g_1)}_{h_1} \cdot \underbrace{\varphi(g_1^{-1})}_{h_1^{-1} \in \varphi(G)}$ ✓

(b) $K = \ker\varphi$ is a subgroup of G

(c) φ is an isomorphism:

$\ker \varphi = \{1_G\}$ injective

$\operatorname{im} \varphi = H$ surjective

Example $\quad \mu_N \cong \mathbb{Z}_N$

$$\varphi : \mathbb{Z}_N \longrightarrow \mu_N$$

$$\bar{r} = r + N\mathbb{Z} \longmapsto e^{i\frac{2\pi}{N}r'} \qquad r' \in r + N\mathbb{Z}.$$

isomorphism

① $\varphi(\bar{r}_1 + \bar{r}_2) = \varphi(\bar{r}_1)\cdot\varphi(\bar{r}_2)$ ✓ homo.

$\cdot_{\mathbb{Z}_N}$ $\qquad \cdot_{\mu_N}$

② $\varphi(\bar{r}) = 1 \Leftrightarrow \bar{r} = \bar{0}$ ✓ inj

③ $\forall\, \omega^j \in \mu_N.\ \exists\, \varphi(\bar{r}_j) = \omega^j$ ✓ suj.

Example. P_k power map

$$P_k : \mu_N \longrightarrow \mu_N$$

$$z \longmapsto z^k$$

① $(z_1 \cdot z_2)^k = z_1^k \cdot z_2^k$ hom.

② isomorphism. $\gcd(k, N) = 1$?

$k = N\mathbb{Z} \qquad P_k(z) = 1$ trivial

$\mu_4 \longrightarrow \mu_4 \qquad k = 2$

$$\ker(P_2) = \{\pm 1\}$$

$$\operatorname{im}(P_2) = \{\pm 1\}$$

$\cong \mathbb{Z}_2$

③ $U(1) \to SU(2)$

$$\varphi(z) := \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix}$$

$$\ker(\varphi) = \{ z \in U(1) \mid z^N = 1 \} \cong \mu_N.$$

$$\mathrm{im}\, \varphi \cong U(1)$$

__Example__ $U(1) \cong SO(2,\mathbb{R})$ $\quad z = e^{i\phi} \to e^{-J\phi} \in SO(2)$

__Example__ $SU(2) \leftrightarrow SO(3)$

Physics context: __spin rotations__

Pauli matrices σ^α $(\alpha = 1,2,3 \ / x.y.z)$ $\quad S^\alpha = \frac{1}{2}\sigma^\alpha$

$$S^{(3)} |\varphi_\pm\rangle = \pm\frac{1}{2} |\varphi_\pm\rangle$$

rotation around $x.y.z$ $\quad \hat{U}_\theta^{(\alpha)} = \exp(-i\theta S^\alpha)$

$$\hat{U}_\theta^3 = \begin{pmatrix} e^{-\frac{i}{2}\theta} & 0 \\ 0 & e^{\frac{i}{2}\theta} \end{pmatrix} \qquad \hat{U}^1 = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$\hat{U}^2 = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

rotated WF: $|\psi'\rangle = U_\theta^{(\alpha)} |\psi\rangle$. exp. value of S^α

$$\langle\psi | S^\alpha | \psi\rangle \to \langle\psi' | \underline{S^\alpha} | \psi'\rangle = \langle\psi | \underline{\underline{U^+ S^\alpha U}} | \psi\rangle$$

$$\begin{pmatrix} U^{\alpha+} \hat{S}^{(1)} U^\alpha \\ U^{\alpha+} \hat{S}^{(2)} U^\alpha \\ U^{\alpha+} \hat{S}^{(3)} U^\alpha \end{pmatrix} = R_\theta^{(\alpha)} \begin{pmatrix} \hat{S}^{(1)} \\ \hat{S}^{(2)} \\ \hat{S}^{(3)} \end{pmatrix} \qquad U_\theta^\alpha \Leftrightarrow R_\theta^\alpha \, ?$$

Example $SU(2) \leftrightarrow SO(3)$

$$\left(u\, \vec{x}\cdot\vec{\sigma}\, u^{-1} = (R(u)\vec{x})\cdot\vec{\sigma} \right)$$

$$\vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$$

$$\vec{x}\cdot\vec{\sigma} = x_1\sigma^1 + x_2\sigma^2 + x_3\sigma^3$$

① $R^3 \to 2\times 2$?

Def . homomorphism

$h : R^3 \to \mathcal{H}_2^0$ (vector space of 2×2 traceless matrices)

$$h(\vec{x}) = \vec{x}\cdot\vec{\sigma} = x_i \cdot \sigma^i = \begin{pmatrix} x^3 & x^1 - i x^2 \\ x^1 + i x^2 & -x^3 \end{pmatrix} \in \mathcal{H}_2^0$$

$$\det(\vec{x}\cdot\vec{\sigma}) = -\vec{x}^2$$

is an isomorphism.

② For a given $u \in SU(2)$. define homomorphism by conjugation:

$$C_u : \mathcal{H}_2^0 \to \mathcal{H}_2^0$$

$$C_u(m) := u m u^{-1} \quad (m \in \mathcal{H}_2^0)$$

$$\left(\begin{array}{l} tr(umu^{-1}) = tr(m) = 0 \\ (umu^{-1})^+ = u m^+ u^{-1} = umu^{-1} \\ \Rightarrow C_u(m) \in \mathcal{H}_2^0 \end{array} \right)$$

Define $R(u) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. s.t.

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\boxed{R(u)}"] \arrow[d, "h"'] & \mathbb{R}^3 \arrow[d, "h"] \\ \mathcal{H}_2^0 \arrow[r, "C_u"'] & \mathcal{H}_2^0 \end{tikzcd}$$

$$h \circ \underline{R(u)} = C_u \circ h$$

$$(R(u)\cdot\vec{x})\cdot\vec{\sigma} = u\,\vec{x}\cdot\vec{\sigma}\,u^{-1}$$

$$(\vec{x} \in \mathbb{R}^3)$$

In other words. we define a homomorphism

$$R : SU(2) \longrightarrow GL(3, \mathbb{R})$$

s.t. $\forall \vec{x} \in \mathbb{R}^3$. $R(u)$ satisfy

$$\boxed{u\,\vec{x}\cdot\vec{\sigma}\cdot u^{-1} = (R(u)\,\vec{x})\cdot\vec{\sigma}}$$

$$u x_i \sigma^i u^{-1} = (R(u)_{ji}\, x_i)\cdot \sigma_j \qquad \forall \vec{x} \in \mathbb{R}$$

$$\Rightarrow u \sigma^i u^{-1} = R(u)_{ji}\, \sigma_j$$

$$(u_1 u_2)\, \sigma_i\, (u_1 u_2)^{+} = u_1 (R(u_2)_{ji}\, \sigma_j)\, u_1^{+}$$

$$= R(u_1)_{ji} (u_1 \sigma_j u_1^{+})$$

$$= \underline{R(u_2)_{ji}\, R_{kj}(u_1)\, \sigma_k}$$

$$= R(u_1 u_2)_{ki}\, \sigma_k$$

$$\Rightarrow R(u_1 u_2) = R(u_1)\cdot R(u_2)$$

① $\vec{y} = R(u)\cdot\vec{x} \qquad \det(\vec{x}\cdot\vec{\sigma}) = -\vec{x}^{\,2}$

$$\vec{y}^{\,2} = -\det((R(u)\cdot\vec{x})\cdot\vec{\sigma}) = -\det(u\,\vec{x}\cdot\vec{\sigma}\,u^{-1}) = \vec{x}^{\,2}$$

$$\Rightarrow \quad R(u) \in O(3)$$

② $$R(\mathbb{1}_2 \in SU(2)) = \mathbb{1}_3 \qquad R(u) \overset{?}{\in} SO(3)$$

$$\mathrm{tr}(\sigma^i \sigma^j \sigma^k) = \epsilon_{ijk} \cdot (2i)$$

$$2i = \mathrm{tr}(\sigma^1 \underbrace{\;}_{uu^\dagger} \sigma^2 \underbrace{\;}_{uu^\dagger} \sigma^3 \overbrace{\;}^{uu^\dagger}) = \mathrm{tr}(\underline{u\sigma^1 u^\dagger}\, \underline{u\sigma^2 u^\dagger}\, \underline{u\sigma^3 u^\dagger})$$

$$= R_{i1}(u) R_{j2}(u) R_{k3}(u)\, \underline{\mathrm{tr}(\sigma^i \sigma^j \sigma^k)}$$

$$= (2i) \cdot \underline{\epsilon_{ijk} R_{i1}(u) R_{j2}(u) R_{k3}(u)}$$

$$= (2i)[\det R(u)]$$

$$\Rightarrow \det R(u) = 1$$

$$\Rightarrow \quad R(u) \in SO(3)$$

$R(u) = R(-u)$ $\qquad$ $SU(2)$ double cover of $SO(3)$

$$\underline{\mathrm{Ker}\, R = \{\pm 1\}} \cong \mathbb{Z}_2$$

Example. $GL(V)$ and $GL(n, K)$

Let $GL(V) : V \to V$ be the group of invertible linear transformations with a finite dimensional vector space V.

Given an ordered basis $b = \{\hat{e}_1, \dots \hat{e}_n\}$

Define a homomorphism:

$$\varphi_b : GL(V) \longrightarrow GL(n, K)$$

$$\tau \longmapsto T_b(\tau)$$

$$\text{s.t.} \quad \underline{\tau(\hat{e}_i) = \sum_j \hat{e}_j \cdot T_b(\tau)_{ji}} \quad @$$

$$\forall \vec{v} \in V. \quad \vec{v} = \sum_{i=1}^{n} v_i \hat{e}_i \qquad (v_i \in K)$$

$$\tau \vec{v} = \sum_{i=1}^{n} v_i (\tau \hat{e}_i) = \sum_{ij} \hat{e}_j \cdot T_b(\tau)_{ji} v_i$$

$$\Rightarrow \tau_1(\tau_2 \vec{v}) = \sum_{ij} (\tau_1 \hat{e}_j) T_b(\tau_2)_{ji} v_i$$

$$= \sum_{ijk} \hat{e}_k \cdot T_b(\tau_1)_{kj} T_b(\tau_2)_{ji} v_i$$

$$= \sum_{ik} \hat{e}_k \underline{[T_b(\tau_1) T_b(\tau_2)]_{ki}} v_i$$

$$\equiv (\tau_1 \tau_2) \vec{v}$$

$$= \sum_{ik} \hat{e}_k \underline{[T_b(\tau_1 \tau_2)]_{ki}} v_i$$

$$\Rightarrow T_b(\tau_1 \tau_2) = T_b(\tau_1) T_b(\tau_2)$$