Problem 22 (Haar measure of SU(2))[1]

Recall that an element of $SU(2)$ $g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ can be expressed using Euler angles with $\alpha = e^{\mathrm{i}\frac{1}{2}(\phi+\psi)}\cos\theta/2$ and $\beta = \mathrm{i}e^{\mathrm{i}\frac{1}{2}(\phi-\psi)}\sin\theta/2$.

(a) Show that

$$\int_{SU(2)} dg g_{\alpha\beta} = 0$$

$$\int_{SU(2)} dg g_{\alpha\beta} g_{\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}.$$

ϵ here is the Levi-Civita / totally antisymmetric tensor. In 2D, $\epsilon = \mathrm{i}\sigma_2$.

(b) Using left and right-invariance show that

$$\int_{SU(2)} dg g_{\alpha_1\beta_1}\cdots g_{\alpha_n\beta_n}$$

can be nonzero if n is even and half of the α's (β's) are 1.

Problem 23

Find three different real irreps for S_3. (*Hint:* two are one dimensional and one two dimensional.)

Problem 24

Recall from the lecture the decomposition of the canonical representation of S_n, $\mathbb{R}^n = W \oplus W^\perp$, where W is the one-dimensional invariant subspace spanned by the vector $v = \sum_{i=1}^n \hat{e}_i$ and $W^\perp$ is the orthogonal complement of W. Show that $W^\perp$ is irreducible.

[1] pp.236-237 of [GM]