Problem 19 (Real representation) [1]

Let V be a complex vector space. The complex conjugate representation sends $g \to \bar{T}(g) \in GL(\bar{V})$. If $\{v_i\}$ is an ordered basis for V then there is a canonical ordered basis $\{\bar{v}_i\}$ for $\bar{V}$.

A *real representation* is one where $(\bar{T}, \bar{V})$ is equivalent to (T, V). Show that for a real representation T, there exists $S \in GL(n, \mathbb{C})$ s.t. $\forall g \in G$,

$$M_T^*(g) = S M_T(g) S^{-1}$$

Problem 20 [2]

Let $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$ be representation spaces for G. The vector space of linear transformations from V to W, denoted $Hom(V, W)$ is also a representation, via the identification $Hom(V, W) = V^* \otimes W$. Define

$$(\tilde{T}(g) \cdot \phi)(v) = T_W(g) \cdot \phi(T_V(g^{-1})v)$$

for $\phi \in Hom(V, W)$ and $v \in V$.

(1) Verify that it is indeed a representation.

(2) Show that the dual representation is a special case of this representation.

(3) Equip V with basis $\{e_i\}$ and W with basis $\{e_a\}$. Identify $Hom(V, W) \cong Mat_{m\times n}(\mathbb{C})$, show that

$$\tilde{T}(g) e_{ai} = M_{ba}(g)(M(g)^{tr,-1})_{ki} e_{bk},$$

where e_{ij} are the unit matrices (1 at $\{i, j\}$, 0 otherwise).

Problem 21

For a representation (T, V) of a *finite group* G on an inner product space V. Define an inner product s.t.

$$\langle T(g)v, T(g)w\rangle = \langle v, w\rangle, \quad \forall v, w \in V, \forall g \in G.$$

[1] p.228 of [GM]
[2] p.229 of [GM]