P7. Quaternion $\to$ V.

There are many homomorphisms.

One example:

$Q = \{\pm 1, \pm i, \pm j, \pm k\}$ $\quad V = \{1, a, b, ab\}$

$\varphi : Q \to V$

define $\varphi(i) = a$. $\quad \varphi(j) = b \quad \varphi(k) = \varphi(ij) = \varphi(i)\varphi(j) = ab$

$\varphi(-1) = \varphi(i)\varphi(i) = a^2 = 1$

$\varphi(-i) = \varphi(i)\varphi(-1) = a$

$\varphi(-j) = b$, $\quad \varphi(-k) = ab$

$\ker \varphi = \{\pm 1\} \cong \mathbb{Z}_2$

$\operatorname{im} \varphi = V$

map the generators!

P8.

$$\begin{tikzcd} \mathbb{Z}_N \arrow[r, "m_{k_1}"] \arrow[d, "\varphi"] & \mathbb{Z}_N \arrow[d, "\varphi"] \\ \mu_N \arrow[r, "P_{k_2}"] & \mu_N \end{tikzcd}$$

commutes $\iff$ $k_1 = k_2 \bmod N$

① $\Leftarrow$ trivial.

② $\Rightarrow$ if $k_1 \neq k_2 \bmod N$

$\hookrightarrow$: $P_{k_2}(\varphi(\bar{i})) = P_{k_2}(\omega^{i+nN}) = \omega^{ik_2 \bmod N}$

$\rightarrow$: $\varphi(m_{k_1}(\bar{i})) = \varphi(\overline{k_1 i}) = \omega^{ik_1 \bmod N}$

$\forall i$, $ik_1 = ik_2 \bmod N$

$k_1 = k_2 \bmod N$.