## 8.16. Representations of $SU(2)$. (§11.18 Moore) sans induced rep.

### 8.16.1. Homogeneous polynomials

$\forall g \in SU(2)$. $g = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}$ $|u|^2+|v|^2=1$. $u, v \in \mathbb{C}$.

Consider $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$. $\in L^2(\mathbb{C}^2)$

$$(u, v) \longmapsto f(u, v).$$

group action $g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$

$$g \cdot f(u, v) = f\left[\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}^{-1} \cdot \begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$$= f\left[\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$$= f(\bar{\alpha}u + \bar{\beta}v, -\beta u + \alpha v)$$

For $\{f_i\}$ to be a rep of $SU(2)$:

$$g \cdot f_i = \sum_j D(g)_{ji} f_j$$

We take $\{f_i\}$ to be homogeneous polynomials in $u, v$ of degree $2j$ ($u^{j+m} v^{j-m}$, $m = -j, \cdots j$)

$\dim \mathcal{H}_{2j} = 2j+1$. Why? this is the $\mathrm{Sym}^{2j}(\mathbb{C}^2)$ (Schur-Weyl.) $\downarrow$ irreps

$$(g \cdot \tilde{f}_{j,m})(u, v) = \hat{f}_{j,m}(\bar{\alpha}u + \bar{\beta}v, -\beta u + \alpha v)$$

$$= (\bar{\alpha}u + \bar{\beta}v)^{j+m}(-\beta u + \alpha v)^{j-m}$$

$$:= \sum_{m'} D^j_{m'm}(g)\, \hat{f}_{j,m'}$$

$$\beta = 0: \quad g\cdot \tilde{f}_{j,m} = \bar{\alpha}^{j+m}\alpha^{j-m} \tilde{f}_{j,m} = \alpha^{-2m}\tilde{f}_{j,m}$$

$$\tilde{D}^{j}_{m'm} = \alpha^{-2m}\delta_{m'm}$$

$$g = e^{-i\sigma^3\phi} = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \Rightarrow g\cdot \tilde{f}_{j,m} = e^{i2m\phi}\tilde{f}_{j,m} \propto |j,m\rangle$$

In QM: angular momentum states $|j,m\rangle$ $\quad J_z|j,m\rangle = \hbar m|j,m\rangle$

$$e^{-iJ_z\phi}|j,m\rangle = e^{-i\hbar m\phi}|j,m\rangle$$

$$\text{LHS} = \binom{j+m}{s}\bar{\alpha}^s\bar{\beta}^{j+m-s}(u^s v^{j+m-s})\binom{j-m}{t}(-\beta)^t\alpha^{j-m-t}\, u^t v^{j-m-t}$$

$$= \sum_{\substack{s,t \\ (s,t\geq 0)}} \binom{j+m}{s}\binom{j-m}{t}\bar{\alpha}^s\alpha^{j-m-t}\bar{\beta}^{j+m-s}(-\beta)^t u^{s+t}v^{2j-s-t}$$

$$\Rightarrow \tilde{D}^{j}_{m'm}(g) = \sum_{s+t=j+m'}\binom{j+m}{s}\binom{j-m}{t}\bar{\alpha}^s\alpha^{j-m-t}\bar{\beta}^{j+m-s}(-\beta)^t$$

$$j = \tfrac{1}{2} \qquad \tilde{D}^{\frac{1}{2}}(g) = \begin{array}{cc} \tfrac{1}{2} & -\tfrac{1}{2} \end{array} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} = g^*$$

$$m = 1/2 \quad \sum_{\substack{s+t \\ = j+m'}} \binom{1}{s}\binom{0}{\underset{t=0}{0}}\bar{\alpha}^s\alpha^0\bar{\beta}^{1-s}(-\beta)^0$$

$$m' = \tfrac{1}{2} \quad s=1 \qquad m' = -\tfrac{1}{2} \quad s=0$$

$$m = -1/2 \quad \sum_{t=\frac{1}{2}+m'} \binom{0}{0}\binom{1}{t}\bar{\alpha}^0\alpha^{1-t}\bar{\beta}^0(-\beta)^t$$

$$m' = \tfrac{1}{2} \quad t=1 \qquad m' = -\tfrac{1}{2} \quad t=0$$

Remark: reps of $SO(3)$

$SO(3) \cong SU(2)/\mathbb{Z}_2$

( recall the homomorphism $\pi : SU(2) \to SO(3)$

$u \vec{x}\cdot\vec{\sigma}\, u^{-1} = (\pi(u)\vec{x})\cdot\vec{\sigma}$

with $\pi(u) = \pi(-u)$

central extension $1 \to \mathbb{Z}_2 \xrightarrow{\iota} SU(2) \xrightarrow{\pi} SO(3) \to 1$ )

we can then obtain the irreps of $SO(3)$

Note that $\tilde{D}^j_{m'm} = \delta_{m'm}\, \alpha^{-2m}$ for diagonal matrices. so $g = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ acts on $V_j$ as

$(-1)^{-2m}\, \mathbb{1}_{V_j}$

which should act trivially. for all $m$.
then $m$ has to be integer. so does $j$.
( $\tilde{D}^j \to SO(3)$ ker $= \mathbb{1}$ iff $j =$ integer )

So the irreps of $SO(3)$ are given by

$V_j$ with $j \in \mathbb{Z}$. and thus $\dim_{\mathbb{C}} V_j = 2j+1$ odd.

## 8.16.2 Characters and irreducibility

① Are $H_{2j}$ reducible? $\Rightarrow$ $\langle \chi_j , \chi_{j'} \rangle \overset{?}{=} \delta_{jj'}$

$g \sim d(z) = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$ $\quad z = e^{i\theta}$ labels a conj. class

( or equivalently $g = \cos\theta \, \mathbb{1} + i \sin\theta \, (\hat{n} \cdot \hat{\sigma})$ )

$$\tilde{D}^j_{m'm}(d(z)) = \sum_{\substack{s+t=j+m' \\ t=0 \\ j+m=s}} \bar{\alpha}^{j+m} \alpha^{j-m} = z^{-2m} \delta_{mm'}$$

Chebyshev polynomial 2nd kind $U_{2j}(\cos\theta)$ $\parallel$ $\frac{\sin(2j+1)\theta}{\sin\theta}$ =

$$\tilde{D}^j(d(z)) = \mathrm{diag}\{ z^{-2j}, z^{-2j+2}, \dots, z^{2j} \}$$

$$\chi_j(g) = z^{-2j}(1 + z^2 + \dots + z^{4j}) = \frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}} \quad (z = e^{i\theta})$$

Haar measure: $\quad dg = \frac{1}{2\pi^2} \sin^2\theta \, d\theta \, \underbrace{d\Omega(\hat{n})}_{4\pi}$

$\theta \in [0, \pi)$

$$4\pi \times \frac{1}{2\pi^2} \int_0^\pi f(\theta) \sin^2\theta \, d\theta = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin^2\theta \, d\theta = -\frac{1}{4\pi i} \oint f(z) (z - z^{-1})^2 \frac{dz}{z}$$

$\left( f(\theta) = \overline{\chi_j} \chi_{j'} \text{ . even in } \theta \right)$ $\quad = \frac{1}{4\pi i} \oint f(z) \, \overline{(z - z^{-1})} (z - z^{-1}) \frac{dz}{z}$

$$\langle \chi_j , \chi_{j'} \rangle = \frac{1}{4\pi i} \oint \underbrace{\overline{(z^{2j+1} - z^{-2j-1})}}_{z^{-\ell}} (z^{2j'+1} - z^{-2j'-1}) \frac{dz}{z}$$

$\ell = 2j+1$

$$= \frac{1}{4\pi i} \oint ( \underbrace{z^{\ell'-\ell-1}}_{2\pi i \delta_{\ell\ell'}} - z^{-\ell-\ell'-1} - z^{\ell+\ell'-1} + \underbrace{z^{\ell-\ell'-1}}_{2\pi i \delta_{\ell\ell'}} ) \, dz$$

$$= \delta_{jj'}$$

② check the self-intertwiner $A$.

$$A\hat{D} - \tilde{D}A = 0$$

$$\sum_n A_{mn} \tilde{D}^j_{nl} = \sum_n \tilde{D}^j_{mn} A_{nl}$$

$$\sum_n A_{mn} z^{-2n} \delta_{nl} = \sum_n z^{-2m} \delta_{mn} A_{nl}$$

$$A_{ml} z^{-2l} = z^{-2m} A_{ml}$$

$$A_{ml} (z^{-2l} - z^{-2m}) = 0 \qquad \forall z.$$

$$\Rightarrow A_{ml} = a_m \delta_{ml}.$$

For arbitrary $\tilde{D}$. $(A\tilde{D})_{ml} = (\tilde{D}A)_{ml}$ $\Rightarrow A_{mm} = A_{ll}$.

$$A_{mm} \tilde{D}_{ml} = \tilde{D}_{ml} A_{ll}$$

$\Rightarrow A = a \cdot \mathbb{1}_{2j}$ is the only possible self-intertwiner.

Shur's lemma $\Rightarrow$ irrep.

## 8.16.3 Unitarization

$$\tilde{f}^m_{2j} = u^{j+m} v^{j-m}, \quad u, v \in \mathbb{C}^2 \qquad \bar{f} f \text{ diverges}$$

$$\langle f_1, f_2 \rangle_{H_{2j}} = \frac{1}{\pi (2j+1)!} \int_{\mathbb{C}^2} \overline{f_1(u,v)} f_2(u,v) e^{-(|u|^2+|v|^2)} d^2u \, d^2v$$

$$\langle g f_1, g f_2 \rangle_{H_{2j}} \overset{g \text{ unitary}}{=} \langle f_1, f_2 \rangle_{H_{2j}}$$

$$\hookrightarrow \quad f_{j,m} = \frac{1}{\sqrt{\pi}} \sqrt{\frac{(2j+1)!}{(j+m)!(j-m)!}} u^{j+m} v^{j-m}$$

$$\tilde{D}^j_{m'm}(g) = \sum_{s+t=j+m'} \binom{j+m}{s}\binom{j-m}{t} \bar{\alpha}^s \alpha^{j-m-t} \bar{\beta}^{j+m-s} (-\beta)^t$$

$$\begin{cases} g = e^{i\frac{\phi}{2}\sigma^3} e^{i\frac{\theta}{2}\sigma^2} e^{i\frac{\psi}{2}\sigma^3} & \phi \in [0, 2\pi) \\ & \theta \in [0, \pi) \\ & \psi \in [0, 4\pi) \\ = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \quad s = j+m'-t & t = j+m'-s \\ \alpha = e^{i\frac{1}{2}(\psi+\phi)} \cos\frac{\theta}{2} \quad \beta = -e^{i\frac{1}{2}(\psi-\phi)} \sin\frac{\theta}{2} \end{cases}$$

$$\hookrightarrow \tilde{D}_{m'm}(g) = \sum_t \binom{j+m}{j+m'-t}\binom{j-m}{t} e^{i\frac{1}{2}(\psi+\phi)[j-m-t-(j+m'-t)]}$$

$s = j+m'-t$

$$(-1)^{j+m-(j+m'-t)} \times e^{i\frac{1}{2}(\psi-\phi)[t-j-m+j+m'-t]}$$

$$\left(\cos\frac{\theta}{2}\right)^{j-m-t+(j+m'-t)} \left(\sin\frac{\theta}{2}\right)^{t+j+m-(j+m'-t)}$$

$$= (j+m)!\,(j-m)!\, e^{-i(m\psi+m'\phi)} \times \sum_t (-1)^{t+m-m'} \frac{\left(\cos\frac{\theta}{2}\right)^{2j-m-m'-2t} \left(\sin\frac{\theta}{2}\right)^{m-m'+2t}}{(j+m'-t)!\,(m-m'+t)!\,(j-m-t)!\,t!}$$

$$= \underbrace{e^{-im'\phi} d^j_{m'm}(\theta) e^{-im\psi}}_{\text{Wigner D-matrix}} \sqrt{\frac{(j+m)!\,(j-m)!}{(j+m')!\,(j-m')!}}$$

in physics.. $D^j_{m'm} = \langle j, m' | \exp\left(\frac{-\hat{J}\cdot\hat{n}\phi}{\hbar}\right) | j, m\rangle$

$\Rightarrow$ It is clear that $D^j$ is a unitary matrix in the $|j m\rangle$ basis.

$$[(D^j)^\dagger D^j]_{mm'} = \sum_k \overline{D^j_{km}} D^j_{km'} = \delta_{mm'} \qquad (D^j D^{j\dagger})_{mm'} = \sum_k D^j_{mk} \overline{D^j_{m'k}} = \delta_{mm'}$$

in addition. $\displaystyle\int_{SU(2)} dg\, \overline{D^{j'}_{m'k'}} D^j_{mk} = \frac{8\pi^2}{2j+1} \delta_{m'm}\delta_{k'k}\delta_{j'j}$

## §16.4 The Clebsch-Gordan decomposition of SU(2)

(§11.20 Moore) (Tinkham. GT & QM book)

Now consider $V_{j_1} \otimes V_{j_2}$. decompose using character theory:

$$\chi_j(z) = \frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}}$$

$j_1 = \frac{1}{2}$. then $\chi_{1/2} = z + z^{-1}$

$$\chi_{\frac{1}{2}\otimes j} = \chi_{1/2}\chi_j = (z + z^{-1})\frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}}$$

$$= \frac{(z^{2j+2} - z^{-2j-2}) + (z^{2j} - z^{-2j})}{z - z^{-1}}$$

$$= \chi_{j+\frac{1}{2}} + \chi_{j-\frac{1}{2}}$$

$$\Rightarrow V_{\frac{1}{2}} \otimes V_j \cong V_{j+\frac{1}{2}} \oplus V_{j-\frac{1}{2}}$$

in general

$$\chi_{j_1\otimes j_2} = \chi_{j_1} \cdot \chi_{j_2} = \frac{z^{2j_1+1} - z^{-2j_1-1}}{z - z^{-1}} \cdot \frac{z^{2j_2+1} - z^{-2j_2-1}}{z - z^{-1}}$$

$$= \sum_{j=j_1-j_2}^{j_1+j_2} \frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}} \qquad (j_1 \geq j_2)$$

$$= \sum_{j=j_1-j_2}^{j_1+j_2} \chi_j$$

or equivalently

$$\langle \chi_{j_1}\chi_{j_2}, \chi_j \rangle = \begin{cases} 1 & j_1 - j_2 \leq j \leq j_1 + j_2 \\ 0 & \end{cases} \qquad (\langle \chi_j, \chi_{j'} \rangle = \delta_{jj'})$$

$$\Rightarrow V_{j_1} \otimes V_{j_2} \cong \bigoplus_{j=j_1-j_2}^{j_1+j_2} V_j$$

## Clebsch-Gordan coefficient:

$\{\psi_{j,m}\}$ an orthonormal basis set of $V_j$

$P_j$ a projector from $V_{j_1} \otimes V_{j_2}$ onto $V_j$

$\langle jm, P_j(\psi_{j_1 m_1} \otimes \psi_{j_2 m_2})\rangle$ – CG. coefficient.

$\langle jm | j_1 m_1; j_2 m_2\rangle$ in physics, often expressed as "Wigner-3j" symbols:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1 j_2 m_2 | j_3 (-m_3)\rangle$$

$$\left( \begin{array}{l} m_1+m_2+m_3=0 : \text{conservation of } m \\ j_i+j_j \geq j_k : \text{triangular relation} \end{array} \right)$$

$$|jm\rangle = \sum_{m_1, m_2} |j_1 m_1; j_2 m_2\rangle \underbrace{\langle j_1 m_1; j_2 m_2 | jm\rangle}_{\text{CG - coeff.}}$$

trivial rep: $P = \int_G T(g)\, dg \qquad \alpha, \beta \in \{+,-\}$

$$T(g)\,|\alpha\rangle \otimes |\beta\rangle = \sum_{\gamma, \delta} g_{\gamma\alpha}\, g_{\delta\beta}\, |\gamma\rangle \otimes |\delta\rangle$$

HW07: $\int_{SU(2)} dg\, g_{\alpha\beta}\, g_{\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}$

$$\Rightarrow P = \frac{1}{2} \epsilon_{\gamma\delta} \epsilon_{\alpha\beta}$$

$$\Rightarrow P|\alpha\rangle \otimes |\beta\rangle = \sum_{\gamma, \delta} \frac{1}{2} \epsilon_{\gamma\delta} \epsilon_{\alpha\beta} |\gamma\rangle \otimes |\delta\rangle$$

$$= \frac{1}{2} \epsilon_{\alpha\beta} (|+,-\rangle - |-,+\rangle)$$

$$\psi_s = \frac{1}{\sqrt{2}} (|+\rangle|-\rangle - |-\rangle|+\rangle)$$

CG: $\langle 0,0 | \tfrac{1}{2}, \pm\tfrac{1}{2}; \tfrac{1}{2}, \mp\tfrac{1}{2}\rangle = \pm\frac{1}{\sqrt{2}}$ otherwise 0

Connection to Wigner D-matrices: $g \in SU(2)/SO(3)$

$$g \cdot \psi_{j_1 m_1} = \sum_{m_1'} D^{j_1}_{m_1' m_1}(g) \psi_{j_1 m_1'}$$

$\psi_{jm}$. ON basis of irrep $j$.

$$g \cdot \psi_{j_2 m_2} = \sum_{m_2'} D^{j_2}_{m_2' m_2}(g) \psi_{j_2 m_2'}$$

recall $g(\psi_{j_1 m_1} \otimes \psi_{j_2 m_2}) = g\psi_{j_1 m_1} \otimes g\psi_{j_2 m_2}$

$$= \sum_{m_1' m_2'} \underbrace{D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2}}_{(D^{j_1} \otimes D^{j_2})_{m_1' m_2', m_1 m_2}} \psi_{j_1 m_1'} \psi_{j_2 m_2'}$$

$D^{j_1} \otimes D^{j_2} \cong \bigoplus_{|j_1 - j_2|}^{j_1+j_2} D^j \rightarrow$ labeled in $JM$.

$$= A^{-1} M(g) A \qquad M(g) = \delta_{jj'} D_{m'm}$$

$$D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2} = \sum_{j, m, m'} A^{-1}{}_{m_1' m_2', j m'} D^j_{m' m} A_{jm, m_1 m_2}$$

$$\psi^j_m = \sum_{m_1 m_2} \psi^{j_1}_{m_1} \psi^{j_2}_{m_2} (A^{-1})_{m_1 m_2 j m} \quad \text{or}$$

$$\psi^{j_1}_{m_1} \psi^{j_2}_{m_2} = \sum_{JM} \psi^j_m A_{jm, m_1 m_2}$$

A transforming between two ON basis $\Rightarrow$ unitary

$$A_{JM, m_1 m_2} = \langle \psi^j_m | \psi^{j_1}_{m_1} \psi^{j_2}_{m_2} \rangle \quad \text{CG-coefficients}$$

$$D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2} = \sum_{|j_1 - j_2|}^{j_1+j_2} \sum_{mm'} \langle jm' | j_1 m_1', j_2 m_2' \rangle \langle jm | j_1 m_1, j_2 m_2 \rangle D^j_{m'm}$$

## 8.16.5 Wigner-Eckart theorem.

For systems with <u>rotational symmetry</u> the states transforming following irreps $\underline{\underline{j}}$

$\psi^{\alpha}_{jm}$. where $\alpha$ labels other "quantum numbers", and $m$ indices withing irrep $j$.

➢ How does an operator look like within an irrep?

<u>group action on operators</u>

After rotation. $\hat{O} \to \hat{O}'$. $\psi \to \psi'$, then

$$\hat{O}'\psi' = (\hat{O}\psi)'$$

$$\hat{O}'(g\psi) = g(\hat{O}\psi) = g\hat{O}g^{-1}(g\psi)$$

$$\Rightarrow \hat{O}' = g\hat{O}g^{-1}$$

<u>Irreducible Tensor operators</u>: operators transforming as irreps of rotational group.

$$g\,\hat{O}^{j}_{m}\,g^{-1} \equiv \sum_{m'} D^{j}_{m'm}\,O^{j}_{m'}$$

examples: rank-0: total energy $(\hat{H})$

density $(\hat{N})$

rank-1: angular momentum $J$

: dipole/ position operator $\vec{r}$

$$\langle \psi^{\alpha}_{j_1 m_1} | \hat{O}^j_m | \psi^{\beta}_{j_2 m_2} \rangle = \langle \psi^{\alpha}_{j_1 m_1} | g^{-1} g \underbrace{\hat{O}^j_m g^{-1}} g | \psi^{\beta}_{j_2 m_2} \rangle$$

$$= \langle \sum_{m_1'} \psi^{\alpha}_{j_1 m_1'} D^{j_1}_{m_1' m_1} \Big| \Big( \sum_{m'} D^{j}_{m' m} \hat{O}^j_m \Big) \Big| \sum_{m_2'} \psi^{\beta}_{j_2 m_2'} D^{j}_{m_2' m_2} \rangle$$

$$= \sum_{m_1', m_2', m'} \overline{D^{j_1}_{m_1' m_1}} \underbrace{D^{j}_{m' m} D^{j_2}_{m_2' m_2}} \langle \psi^{\alpha}_{j_1 m_1'} | \hat{O}^j_m | \psi^{\beta}_{j_2 m_2'} \rangle$$

insert $D^{j}_{m'm} D^{j_2}_{m_2' m_2} = \sum_{|j-j_2|}^{j+j_2} \sum_{m_3 m_3'} \langle j_3 m_3' | j\, m'\, j_2 m_2' \rangle \langle j_3 m_3 | j\, m\, j_2 m_2 \rangle D^{j}_{m_3' m_3}$

from above, and use the orthornormal relation

$$\langle \alpha j_1 m_1 | \hat{O}^j_m | \beta j_2 m_2 \rangle = \underbrace{\langle j_1 m_1 | j\, m ; j_2 m_2 \rangle}_{\text{"angular part"}} \times$$

$$\underbrace{\Big( \sum_{m_1' m_2' m'} \langle j_1 m_1' | j\, m' ; j_2 m_2' \rangle \langle \alpha j_1 m_1' | \hat{O}^j_{m'} | \beta j_2 m_2' \rangle \Big)}_{\langle j_1 \| \hat{O}^j \| j_2 \rangle}$$

reduced matrix element.

independent of m's.

"radial part"

$\Rightarrow$ selection rules from the angular part.

often in atomic/spectroscopic contexts.

We will see concrete examples in later lectures.