8.13. (§11.16) Schur–Weyl duality and irreps of $GL(d, K)$

Fulton & Harris Chap 6. Corrections in magenta

$V^{\otimes 2}$ as a representation of S_2: $(V = K^d,\ K = \mathbb{R}, \mathbb{C})$

$$\sigma : v_1 \otimes v_2 \longmapsto v_2 \otimes v_1$$

$$V \otimes V \cong \underline{D^{1+} \otimes \mathbb{1}^{+}} \oplus D^{1-} \otimes \mathbb{1}^{-}$$

$$\begin{cases} \dim D^{1+} = \dfrac{d(d+1)}{2} & \chi_e = d^2. \\ \dim D^{1-} = \dfrac{d(d-1)}{2} & \chi_\sigma = d \end{cases}$$

$$D^{1+} \otimes \mathbb{1}^{+} = \operatorname{span}\{ v_i \otimes v_j + v_j \otimes v_i \}$$

$$=: \operatorname{span}\{ v_i \cdot v_j ,\ i \le j \} = \operatorname{Sym}^2 V$$

$$D^{1-} \otimes \mathbb{1}^{-} = \operatorname{span}\{ v_i \otimes v_j - v_j \otimes v_i \}$$

$$=: \operatorname{span}\{ v_i \wedge v_j ,\ i < j \} = \Lambda^2 V$$

$$\begin{cases} v_i \wedge v_j = - v_j \wedge v_i \\ v_i \wedge v_i = 0 \end{cases}$$

$\boxed{1\,|\,2}$ $c = \underline{e} + (12)$ $v_i \otimes v_j \longmapsto v_i \otimes v_j + v_j \otimes v_i$

$$\underline{c \cdot V^{\otimes 2}} = \operatorname{span}\{ v_i \otimes v_j + v_j \otimes v_i \} = \underline{\operatorname{Sym}^2 V}$$

$\boxed{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}$ $c = e - (12)$

$$c \cdot V^{\otimes 2} = \operatorname{span}\{ v_i \otimes v_j - v_j \otimes v_i \} = \underline{\Lambda^2 V}$$

$$S_{(2)}: V\otimes V \longrightarrow \mathrm{Sym}^2 V$$

$$\ker(\pi) = \{v_i \otimes v_j - v_j \otimes v_i\}$$

$$S_{(1,1)}: V\otimes V \longrightarrow \Lambda^2 V$$

$$\ker(\pi) = \{v_i \otimes v_j + v_j \otimes v_i\}$$

Any elemets $\in V^{\otimes 2}$. can be given by a rank-2 tensor

$$t = \sum_{ij} a_{ij} \, v_i \otimes v_j$$

Then the action of S_2

$$\sigma \cdot t = \sum_{ij} a_{ij} \, v_{\sigma(i)} \otimes v_{\sigma(j)} = \sum_{ij} a_{ij} \, v_j \otimes v_i = \sum_{ij} a_{ji} \, v_i \otimes v_j$$

defines an action on the tensor.

$$(\sigma \cdot a)_{ij} = a_{ji} \qquad (a \in K^{d^2})$$

V a rep. of group G. $V\otimes V$ is a rep.:

$$T(g)^{\otimes 2}(v_1 \otimes v_2) = T(g)v_1 \otimes T(g)v_2$$

$$T(g)\cdot t = \sum_{ij} a_{ij} \left[\underline{T(g)v_i} \otimes T(g)v_j \right]$$

$$= \sum_{\substack{ij \\ kl}} a_{ij} \, M(g)_{ki} \, M(g)_{lj} \; v_k \otimes v_l$$

defines an action on $\underline{a}$.

$$(g\cdot a)_{kl} = \sum_{ij} M(g)_{ki} \, M(g)_{lj} \, a_{ij}$$

The action of G and S_2 commutes on $V^{\otimes n}$

$$g \cdot [\sigma(v_{i_1} \otimes v_{i_2})] = \sigma[g(v_{i_1} \otimes v_{i_2})]$$

$$\left(\begin{aligned} T(g) v_{\sigma(i_1)} \otimes T(g) v_{\sigma(i_2)} &= \sum_{k\ell} M(g)_{k i_2} M(g)_{\ell i_1} v_k \otimes v_\ell \\ \equiv \sigma\Big[\sum_{k\ell} M(g)_{k i_1} M(g)_{k i_2} v_k \otimes v_\ell\Big] &= \sum_{k\ell} M(g)_{k i_1} M(g)_{k i_2} v_\ell \otimes v_k \end{aligned} \right)$$

$\Rightarrow V^{\otimes n}$ is a rep of $G \times S_n$ $\quad (g, \sigma)\, v_{i_1} \otimes \cdots \otimes v_{i_n} = g \sigma \, v \otimes \cdots$

Schur – Weyl duality theorem: (Fulton & Harris for proofs)

$$V^{\otimes n} \cong \bigoplus_\lambda D_\lambda \otimes R_\lambda$$

R_λ are the irreps of S_n

$D_\lambda = \mathrm{Hom}_{S_n}(R_\lambda, V^{\otimes n})$ the degeneracy space.

The representations D_λ are irreducible representations of $GL(d, K)$ (and its subgroups)

All irreps can be found by varying n

Example. Spin-0 and 1 rep of $SU(2)$

Consider $G = SU(2)$ and S_2

$V = \{|+\rangle, |-\rangle\}$

$V^{\otimes 2} = \{ |s_1\rangle \otimes |s_2\rangle , \ s_i \in V \}$ $\dim = 4$

$V^{\otimes 2} \underset{=}{\overset{u}{\cong}} W_1 \otimes \Gamma^+ \oplus W_0 \otimes \Gamma^-$

$$W_1 = Sym^2 V = \{ \underset{|1,1\rangle}{|+\rangle \otimes |+\rangle} , \underset{|1,0\rangle}{\tfrac{1}{\sqrt{2}} (|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle)} , \underset{|1,-1\rangle}{|-\rangle \otimes |-\rangle} \}$$

$$W_0 = \Lambda^2 V = \{ \underset{|0,0\rangle}{\tfrac{1}{\sqrt{2}} (|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)} \}$$

Now consider the group action of $g \in SU(2)$ on V.

$$g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \qquad |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} g|+\rangle = \alpha|+\rangle + \beta|-\rangle \quad (= g_{11}|+\rangle + g_{21}|-\rangle) \\ g|-\rangle = -\bar{\beta}|+\rangle + \bar{\alpha}|-\rangle \end{cases}$$

$$g|1,1\rangle = g|+\rangle \otimes g|+\rangle = \alpha^2 |++\rangle + \alpha\beta(|+-\rangle + |-+\rangle) + \beta^2 |--\rangle$$

$$= \alpha^2 |1,1\rangle + \sqrt{2}\alpha\beta |1,0\rangle + \beta^2 |1,-1\rangle$$

$$g|1,-1\rangle = \bar{\beta}^2 |++\rangle - \bar{\alpha}\bar{\beta}(|+-\rangle + |-+\rangle) + \bar{\alpha}^2 |--\rangle$$

$$= \bar{\beta}^2 |1,1\rangle - \sqrt{2}\bar{\alpha}\bar{\beta} |1,0\rangle + \bar{\alpha}^2 |1,-1\rangle$$

$$g|1,0\rangle = \frac{1}{\sqrt{2}}\left(g|+\rangle \otimes g|-\rangle + g|-\rangle \otimes g|+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(-2\alpha\bar{\beta}\,|++\rangle + (|\alpha|^2-|\beta|^2)(|+-\rangle+|-+\rangle)\right.$$
$$\left. + 2\bar{\alpha}\beta\,|--\rangle\right)$$

$$= -\sqrt{2}\,\alpha\bar{\beta}\,|1,1\rangle + (|\alpha|^2-|\beta|^2)|1,0\rangle + \sqrt{2}\,\bar{\alpha}\beta\,|1,-1\rangle$$

$$D'(g) = \begin{array}{c} \begin{array}{ccc} |1,1\rangle & |1,0\rangle & |1,-1\rangle \end{array} \\ \begin{pmatrix} \alpha^2 & -\sqrt{2}\alpha\bar{\beta} & \bar{\beta}^2 \\ \sqrt{2}\,\alpha\beta & |\alpha|^2-|\beta|^2 & -\sqrt{2}\,\bar{\alpha}\bar{\beta} \\ \beta^2 & \sqrt{2}\,\bar{\alpha}\beta & \bar{\alpha}^2 \end{pmatrix} \end{array}$$ Wigner-D' matrix

For $W_0 = \{\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)\} \equiv |0,0\rangle$

$g|+\rangle = \alpha|+\rangle + \beta|-\rangle$

$g|-\rangle = -\bar{\beta}|+\rangle + \bar{\alpha}|-\rangle$

$$g\cdot|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\alpha|^2|+-\rangle - |\beta|^2|-+\rangle\right.$$
$$\left. - (|\alpha|^2|-+\rangle - |\beta|^2|+-\rangle)\right)$$
$$= \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) = |0,0\rangle \quad \text{trivial.}$$

$\Rightarrow$ Tensors of definite symmetries (obtained via Young symmetrizers) transform as irreps of $GL(d,k)$.

Symmetric powers of the defining rep. of $SU(2)$ are irreps of $SU(2)$?

<u>Example</u>. $V^{\otimes 3} \overset{\text{span}}{=} \{ v_i \otimes v_j \otimes v_k \}$

S_3	$[()]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1

$\chi([()]) = d^3$

$\chi([(12)]) = d^2$ $\qquad \underline{v_i \otimes v_i \otimes v_k}$

$\chi([(123)]) = d$

$a_{1^+} = \langle \chi_{1^+}, \chi \rangle = \frac{1}{6}(d^3 \times 1 + d^2 \times 3 + d \times 2) = \underline{\frac{1}{6} d(d+1)(d+2)}$

$a_{1^-} = \langle \chi_{1^-}, \chi \rangle = \frac{1}{6}(d^3 - 3d^2 + 2d) = \frac{1}{6} d(d-1)(d-2)$

$a_2 = \langle \chi_2, \chi \rangle = \frac{1}{6}(2d^3 - 2d) = \frac{1}{3} d(d+1)(d-1)$

① $\boxed{1|2|3}$ $\qquad C = PQ = e + (12) + (13) + (23) + (123) + (132)$

$C \cdot V^{\otimes 3} = \text{span} \{ \sum_\sigma v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \}$

$= Sym^3 V$

$t = \sum a_{ijk} v_i \otimes v_j \otimes v_k$

$\sigma \cdot t = \sum a_{ijk} v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)}$

$= \sum a_{\sigma^{-1}(i) \sigma^{-1}(j) \sigma^{-1}(k)} \; v_i \otimes v_j \otimes v_k$

$\Rightarrow (\sigma \cdot a)_{ijk} = a_{\sigma^{-1}(i) \sigma^{-1}(j) \sigma^{-1}(k)}$

$\underline{(a_S)_{ijk} = \sum_\sigma a_{\sigma^{-1}(i), \sigma^{-1}(j), \sigma^{-1}(k)} = \sum_\sigma a_{\sigma(i)\sigma(j)\sigma(k)}}$

$\Rightarrow (a_S)_{jik} = (a_S)_{ijk}$

$(\sigma a_S)_{ijk} = (a_S)_{ijk}$

② $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$ $\quad c = e - (12) - (13) - (23) + (123) + (132)$

$$(a_\wedge)_{ijk} = \sum_\sigma \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(i),\, \sigma^{-1}(j),\, \sigma^{-1}(k)}$$

$$\underline{(a_\wedge)_{jik}} = (\tau(ij)\, a_\wedge)_{ijk}$$

$$= \sum_\sigma \tau(ij)\, \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(i),\, \sigma^{-1}(j),\, \sigma^{-1}(k)}$$

$$= \sum_\sigma \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(j),\, \sigma^{-1}(i),\, \sigma^{-1}(k)}$$

$$= \sum_\sigma \mathrm{sgn}(\sigma\, \tau(ij))\, a_{\sigma^{-1}(i),\, \sigma^{-1}(j),\, \sigma^{-1}(k)}$$

$$= -\sum_\sigma \mathrm{sgn}(\sigma)\, a_{\sigma^{-1}(i),\, \sigma^{-1}(j),\, \sigma^{-1}(k)}$$

$$= \underline{-(a_\wedge)_{ijk}}$$

if $d=2$: $\quad i, j, k \in \{1, 2\}$

$$a_{1,1,2} = -a_{1,1,2} = 0$$

$\Rightarrow$ all elements $a_{ijk} = 0$

$\boxed{\text{$V = k^d$. the irrep corresponding to a Young diagram is 0 of d is smaller than the number of rows of the Young diagram.}}$

③ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad c_{(2,1)} = (e+(12))(e-(13)) = e+(12)-(13)-(132)$

$$c_{(2,1)} V^{\otimes 3} = \mathrm{span}\{ v_i \otimes v_j \otimes v_k + v_j \otimes v_i \otimes v_k - v_k \otimes v_j \otimes v_i - v_k \otimes v_i \otimes v_j \}$$

$$(a_2)_{ijk} = a_{ijk} + a_{jik} - a_{kji} - a_{jki}$$

$i \to k \to j$

$$\left(\begin{array}{l} \sigma : v_i \otimes v_j \otimes v_k \to v_{\sigma(i)} \otimes v_{\sigma(j)} \otimes v_{\sigma(k)} \\ a_{ijk} \to a_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)} \end{array} \right)$$

$i \leftarrow k \leftarrow j$

$$\begin{cases} (a_2)_{ijk} + (a_2)_{jki} + (a_2)_{kij} = 0 & - A \\ (a_2)_{ijk} = -(a_2)_{kji} & - B \end{cases}$$

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$: $B \to (a_2)_{ijk} = -(a_2)_{jik}$

1\. $c_{(n)} \cdot V^{\otimes n} = \mathrm{Sym}^n V$ projects to the totally symmetric sector. $\Leftrightarrow$ bosons

$V = \mathbb{K}^d = \mathcal{H}$ (single-particle Hilbert space)

$$\dim \mathrm{Sym}^n V = \binom{n+d-1}{n} \quad \left(\begin{array}{c} v_{i_1} \cdot v_{i_2} \cdots v_{i_n} \\ i_1 \le i_2 \le \cdots \le i_n \end{array} \right)$$

$(i_n \le d)$

$\Updownarrow$

$v_{i_1} \cdots v_{i_n}$

$i_1 < i_2 < \cdots < i_n$

$i_n \le d+n-1$

$n=3 \quad \frac{1}{6} d(d+1)(d+2)$

Consider a collection of $\underline{d}$ bosonic oscillators

$$h = \frac{1}{2}\hbar\omega \{a^+, a\}$$

$$= \hbar\omega(a^+a + \frac{1}{2})$$

$\hbar\omega = 1$. subtract $\frac{1}{2}$

$$H = \sum_j^d a_j^+ a_j$$

$(\beta = 1/k_BT)$

Its partition function:

$$Z = \underline{\left(\sum_{n=0}^{\infty} e^{-\beta n}\right)^d} = \frac{1}{(1-q)^d} \qquad q = e^{-\beta}$$

$$= \sum_{n=0}^{\infty} q^n \underline{\dim(\mathrm{Sym}^n V)}$$

$\dim(\mathrm{Sym}^n V)$ is the degeneracy of eigenstates with total energy n.

2. For fermionic oscillators

$$h = \frac{1}{2}\hbar\omega [a^+, a]$$

$$= \hbar\omega(a^+a - \frac{1}{2})$$

$$H = \sum_j^d a_j^+ a_j$$

$$Z = \left(\sum_{n=0}^{1} e^{-\beta n}\right)^d = (1+q)^d$$

$$= \sum_{n=0}^{d} q^n \underline{\dim(\Lambda^n V)}$$

$$= \binom{n}{d}$$

3. $G = SU(2) \subset GL(2, \mathbb{C})$ irreps

We consider Young diagrams with at most 2 rows.

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & 3 & \cdots & 2k-1 & 2k+1 & \cdots & \cdot & \cdots & 2k+l \\ \hline 2 & 4 & & 2k \\ \cline{1-4} \end{array}$$

$\underbrace{\qquad\qquad}_{k}$ $\underbrace{\qquad\qquad}_{l}$

The corresponding Young symmetrizer:

$$C_T = P_T \underline{\underline{Q_T}}$$

$$\left(v_{i_1} \wedge v_{i_2} := v_{i_1} \otimes v_{i_2} - v_{i_2} \otimes v_{i_1} \right)$$

$$C_T \cdot v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \quad (i_m \in \{1, 2\})$$

$$= P_T \, (\underline{v_{i_1} \wedge v_{i_2}}) \otimes (\underline{v_{i_3} \wedge v_{i_4}}) \otimes \cdots \otimes (v_{i_{2k-1}} \wedge v_{i_{2k}})$$

$$Q_T = \prod_{i=1}^{k} e - (2i-1, 2i) \qquad \otimes v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}}$$

$v_{i_{2j-1}} \wedge v_{i_{2j}} \neq 0$ iff $i_{2j-1} \neq i_{2j}$ $\quad v_1 \wedge v_2$ or $v_2 \wedge v_1$

The non-zero images of C_T is

$$C_T \bigotimes_{j=1}^{n} v_{i_j} = \underline{P_T \left[\bigotimes^{k} (v_1 \wedge v_2) \right]} \otimes v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}}$$

$$= (-1)^m \bigotimes_{i}^{k} (v_1 \wedge v_2) \otimes P_{T'} \left(v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}} \right)$$

$$T' : \underbrace{\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline \end{array}}_{l}$$

$V^{\otimes n}$ as rep of $SU(2)$.

$u \in SU(2)$ acts on $V_1 \wedge V_2$

$$u \cdot (v_1 \wedge v_2) = u(v_1 \otimes v_2 - v_2 \otimes v_1)$$

$$= \sum_{ij} u_{i1} u_{j2} v_i \otimes v_j - \sum_{ij} u_{i2} u_{j1} v_i \otimes v_j$$

$$= (u_{11} u_{12} - u_{12} u_{11}) v_1 \otimes v_1 + (u_{11} u_{22} - u_{12} u_{21}) v_1 \otimes v_2 + (u_{21} u_{12} - u_{22} u_{11}) v_2 \otimes v_1 + (u_{21} u_{22} - u_{22} u_{21}) v_2 \otimes v_2$$

$$= (\det u)\, v_1 \wedge v_2$$

$$u^{\otimes n}\left(C_T \bigotimes_j^n v_{i_j}\right) = (\cancel{\det u})^{k} \overset{1}{} (-1)^{mk} \bigotimes_i (v_1 \wedge v_2) \otimes u^{\otimes l} P_{T'}(v_{i_{2k+1}} \otimes \cdots \otimes v_{i_{2k+l}})$$

$u \in SU(2)$ acts non-trivially only on $\underline{P_{T'}(v_{i_{2k+1}} \otimes \cdots \otimes v_{i_n})}$

$\Rightarrow$ irreps of $SU(2)$ is in one-to-one correspondence with Young diagrams of a <u>single row</u> of l boxes

Dimension of the irrep.

$d = 2$: $\binom{l+d-1}{l} = \binom{l+1}{l} = l+1$

$\mathrm{span}\{ v_{i_1} \otimes \cdots \otimes v_{i_l} \}$

$i_1 \le i_2 \le \cdots \le i_l$

$\dim = l+1$

in physics, $l = 2j$ "spin-j representation of $SU(2)$"

$\Rightarrow$ irreps : $\mathrm{Sym}^l V$. $V \cong \mathbb{C}^2$ the fundamental rep.

$l=0$ scalar / singlet : 0

$l=1$ $(j=\frac{1}{2})$ spin-1/2

(dublet) : $\uparrow$ $\downarrow$

$l=2$ $(j=1)$ triplet $\uparrow\otimes\uparrow$ $\uparrow\otimes\downarrow$ $\downarrow\otimes\downarrow$

$\underline{\underline{Sym^2 V}}$ $\downarrow$ $\downarrow$ $\downarrow$

$|\uparrow\uparrow\rangle$ $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)$ $|\downarrow\downarrow\rangle$

$S=1$ rep of $SU(2)$ (see above)