Recap. define class operators $\hat{C}_i = \sum_{g \in C_i} g$

① $\forall h \in G.\ [h\ C_i] = 0$

$\Rightarrow$ $\hat{C}_i$ are intertwiners on any rep. space

$$\begin{tikzcd} V \arrow[r, "C_i"] \arrow[d, "g"'] & V \arrow[d, "g"] \\ V \arrow[r, "C_i"'] & V \end{tikzcd}$$

restrict to an irrep. then $\hat{C}_i = \lambda_i^\mu \mathbb{1}_{V^\mu} \equiv \sum_{g \in C_i} T^\mu(g)$

take trace.

$$\lambda_i^\mu \cdot n_\mu = m_i\, \chi^\mu([C_i])$$

$$\lambda_i^\mu = \frac{m_i}{n_\mu} \chi^\mu([C_i])$$

② $\hat{C}_i \hat{C}_j = \sum D_{ij}^k \hat{C}_k$. restrict to V^μ. $C_i = \sum_\mu \lambda_i^\mu P_\mu \cong \bigoplus_\mu \lambda_i^\mu \mathbb{1}_{V_\mu}$

then

$$\lambda_i^\mu \lambda_j^\mu = \sum_k [D_i]_{jk} \lambda_k^\mu \qquad \psi^\mu = (\lambda_1^\mu . \lambda_2^\mu . \lambda_3^\mu)^T$$

$$\equiv \lambda_i^\mu \sum_k \delta_{jk} \lambda_k^\mu$$

$$\Rightarrow \sum_k \left([D_i]_{jk} - \lambda_i^\mu \delta_{jk} \right) \lambda_k^\mu \qquad \psi^\mu = (\lambda_1^\mu . \lambda_2^\mu . \lambda_3^\mu)^T$$

$$(D_i - \lambda_i^\mu \mathbb{1})\, \psi^\mu = 0$$

to diagonalize all $\hat{C}_i$: $\underbrace{\sum_i (\overleftrightarrow{D_i}\, y^i}_{L} - \lambda_i^\mu y^i)\, \vec{\psi}^\mu = 0$

After obtaining λ_i^μ : $\chi^\mu = \frac{n_\mu}{m_i} \lambda_i^\mu$ $\qquad \langle \chi^\mu . \chi^\nu \rangle = \delta_{\mu\nu}$.

S_3: $C_1 = e$. $C_2 = (12) + (13) + (23)$. $C_3 = (123) + (132)$

$m_1 = 1$ $m_2 = 3$ $m_3 = 2$

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1 + 3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$

$L_{jk} = \sum D_{ij}^{k} y^i$

$L_{11} = D_{11}^{1} y^1 + D_{21}^{1} y^2 + D_{31}^{1} y^3$

$= y^1 + 0 + 0$

$L_{22} = D_{12}^{2} y^1 + D_{22}^{2} y^2 + D_{32}^{2} y^3$

$= y^1 + 0y^2 + 2y^3$

$$\Rightarrow L = \begin{pmatrix} y^1 & y^2 & y^3 \\ 3y^2 & y^1 + 2y^3 & 3y^2 \\ 2y^3 & 2y^2 & y^1 + y^3 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} y^1 + \begin{pmatrix} & 1 & \\ 3 & & 3 \\ & 2 & \end{pmatrix} y^2 + \begin{pmatrix} & & 1 \\ & 2 & \\ 2 & & 1 \end{pmatrix} y^3$$

$\lambda^{\mu_1} = y^1 + 3y^2 + 2y^1$

$\lambda^{\mu_2} = y^1 - 3y^2 + 2y^3$

$\lambda^{\mu_3} = y^1 + 0y^2 - y^3$

$$\boxed{\lambda_i^{\mu} = \frac{m_i}{n_\mu} \chi^{\mu}([C_i])}$$

$\chi_i^{\mu} = n_\mu \dfrac{\lambda_i^{\mu}}{m_i}$

$m_1 = 1$ $m_2 = 3$ $m_3 = 2$

$$\begin{array}{lccc} & C_1 & C_2 & C_3 \\ \chi_{\mu_1} = n_{\mu_1} (& \frac{1}{1}, & \frac{3}{3}, & \frac{2}{2}) \\ \chi_{\mu_2} = n_{\mu_2} (& \frac{1}{1}, & \frac{-3}{3}, & \frac{2}{2}) \\ \chi_{\mu_3} = n_{\mu_3} (& \frac{1}{1}, & \frac{0}{3}, & -\frac{1}{2}) \end{array}$$

C_2: CSCO-I

↪ normalization:

$$n_{\mu_1} = n_{\mu_2} = 1, \quad n_{\mu_3} = 2 \quad \Leftarrow \quad \begin{cases} \langle \chi_{\mu_1}, \chi_{\mu_1} \rangle = \frac{1}{6} n_{\mu_1}^2 \cdot 6 = 1 \\ \langle \chi_{\mu_2}, \chi_{\mu_2} \rangle = \frac{1}{6} n_{\mu_2}^2 \cdot 6 = 1 \\ \langle \chi_{\mu_3}, \chi_{\mu_3} \rangle = \frac{1}{6} n_{\mu_3}^2 (1 + 0 + \frac{1}{4} \times 2) \end{cases}$$

$= \frac{1}{4} n_{\mu_3}^2 = 1$

Projectors:

$$\hat{C}_i \, \hat{C}_j = \sum_k [D_i]_{jk} C_k$$

$$\hat{C}_i \cdot \phi_\mu = \lambda_i^\mu \phi_\mu \qquad\qquad \phi_\mu = \sum_i \phi_\mu(C_i) C_i = \phi_\mu^i C_i$$

$$\sum_j \phi_\mu^j \hat{C}_i \hat{C}_j = \lambda_i^\mu \sum_k \phi_\mu^k C_k$$

$$\Rightarrow \sum_{jk} \phi_\mu^j [D_i]_{jk} C_k = \lambda_i^\mu \sum_k \phi_\mu^k C_k$$

$$\Rightarrow \sum_k \Big(\sum_j (D_i^T)_{kj} \phi_\mu^j \Big) C_k = \sum_k \lambda_i^\mu \phi_\mu^k \cdot C_k$$

$$\Rightarrow \sum_j (D_i^T)_{kj} \phi_\mu^j = \lambda_i^\mu \phi_\mu^k$$

$$\sum_j (D_i^T - \lambda_i^\mu \delta_{jk}) \phi_\mu^j = 0$$

ϕ_μ are eigenvectors of D_i^T with basis $\{C_1, C_2, C_3\}$

$$D_2^T = \begin{pmatrix} & 3 & \\ 1 & & 2 \\ & 3 & \end{pmatrix}$$

$\lambda_2^{\mu_1} = 3 \qquad \phi_{\mu_1} \propto (1, 1, 1)^T \leftarrow \chi_\mu(i)$

$\lambda_2^{\mu_2} = -3 \qquad \phi_{\mu_2} \propto (1, -1, 1)^T$

$\lambda_2^{\mu_3} = 0 \qquad \phi_{\mu_3} \propto (2, 0, -1)^T$

$$P_{\mu_1} = \alpha_{\mu_1}(C_1 + C_2 + C_3)$$

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1 + 3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$

$$P_{\mu_1}^2 = \alpha_{\mu_1}^2 (C_1^2 + C_2^2 + C_3^2 + 2C_1C_2 + 2C_1C_3 + 2C_2C_3)$$

$$= \alpha_{\mu_1}^2 (\underline{C_1} + \underline{3C_1} + 3C_3 + \underline{2C_1} + C_3 + 2C_2 + 2C_3 + 4C_2)$$

$$= 6\alpha_{\mu_1}^2 (C_1 + C_2 + C_3) = \alpha_\mu (C_1 + C_2 + C_3) \equiv P_{\mu_1}$$

$$\alpha_{\mu_1} = \frac{1}{6}$$

$$P_{\mu_1} = \frac{1}{6}(C_1 + C_2 + C_3)$$

$$P_{\mu_2} = \frac{1}{6}(C_1 - C_2 + C_3)$$

$$P_{\mu_3} = \frac{1}{3}(2C_1 - C_3)$$

$$P_{\mu_1} P_{\mu_2} \propto C_1^2 + C_3^2 + 2C_1C_3 - C_2^2 = C_1 + 2C_1 + C_3 + 2C_3 - (3C_1 + 3C_3) = 0$$

$$P_{\mu_1} P_{\mu_3} \propto (C_1 + C_2 + C_3)(2C_1 - C_3) = 2C_1^2 - C_1C_3 + 2C_1C_2 - C_2C_3 + 2C_1C_3 - C_3^2$$

$$= 2C_1 - C_3 + 2C_2 - 2C_2 + 2C_3 - (2C_1 + C_3) = 0$$

$$\hat{C}_2 P_{\mu_1} = \frac{1}{6}(C_1C_2 + C_2^2 + C_2C_3)$$

$$= \frac{1}{6}(C_2 + 3C_1 + 3C_3 + 2C_2) = 3 \cdot \frac{1}{6}(C_1 + C_2 + C_3)$$

$$\equiv \frac{m_2}{n_{\mu_1}} \chi_{\mu_1}([C_2]) \cdot P_{\mu_1}$$

$$(12) P_{\mu_1} = (12) \cdot \frac{1}{6}(e \pm (12) \pm (23) \pm (13) + (123) + (132))$$

$$(12) P_{\mu_2} = \frac{1}{6}((12) \pm e \pm (123) \pm (132) + (23) + (13))$$

$$= \chi_{\mu_1} \cdot P_{\mu_1}$$

$$= (-1) \cdot P_{\mu_2} = \chi_{\mu_2} \cdot P_{\mu_2}$$

$$T(h) P^{\mu} = \sum_{i,k=1}^{n_\mu} M^{\mu}_{ki}(h) P^{\mu}_{ki}$$

$$T(h) P^{\mu}_{ij} = \sum_{k=1}^{n_\mu} M^{\mu}_{ki}(h) P^{\mu}_{kj}$$

$$\boxed{\{P^{\mu}_{kj},\ k = 1, \dots n_\mu\}}$$

$$P_{\mu_3} = P^{11}_{\mu_3} + P^{22}_{\mu_3} \qquad P^{\mu}_{ij} P^{\mu}_{kl} = \delta_{jk} P^{\mu}_{il}$$

$$\Rightarrow P^{11} \cdot P^{22} = 0 \qquad P^{\mu}_{11} P^{\mu}_{21} = 0$$

what if $P^{11} = e - (13) + (12) - (132)$?

$$P^{22} = e - (12) + (13) - (123)$$

satisfy the orthogonality relation $\begin{cases} P^{11}_{\mu_3} P^{22}_{\mu_3} = \delta_{12} P^{12}_{\mu_3} = 0 \\ P^{11}_{\mu_3} + P^{22}_{\mu_3} = P_{\mu_3} \end{cases}$

in principle. find more commuting operators to lift degeneracies on the group space R_G

S_2 :

	e	(12)
$\underline{1}$	1	1
$-\underline{1}$	1	-1

	C_1	C_2
C_1	C_1	C_2
C_2	C_2	C_1

P^k_{2j} : $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\lambda = \pm 1$

$P_1 = \frac{1}{2}(e + (12))$

$P_2 = \frac{1}{2}(e - (12))$

$G \supset G_1$

$P^{\nu}_{\nu_1} = P^{\nu} P'^{\nu_1}$

$\hookrightarrow \; P^2_{\underset{-1}{1}} = P^2 \underset{P'^{-1}}{P'^{1}} = \frac{1}{6}(2e - (123) - (132))(e \pm (12))$

$= \frac{1}{6}(2e \pm 2(12) - (123) \mp (13) - (132) \mp (23))$

$\begin{cases} P^2 = P'_1 + P'_{-1} \quad \checkmark \\ P'_1 P'_{-1} = 0 \end{cases}$

$C_2 + C'_2$ CSCO-III

$\uparrow \qquad \uparrow$

$S_3 \qquad S_2$

$(12)\, P^2_{\pm 1} = \pm P^2_{\pm}$

§14 Representation of S_n (Miller, book Chap 4)

see also 陈金全
contains all proofs of the statements below.
(11.15 Moore)

Basics of S_n:

$(i_1, i_2, \cdots i_r) \sim (j_1, j_2, \cdots j_r)$

r-cycles are conjugate

S_n irreps are defined by vectors

$\vec{l} = (l_1, l_2 \cdots l_n)$

l_i the number of i-cycles

conj. classes $\Leftrightarrow$ Young diagrams.

Continue of the group algebra perspective

finding irreps = finding (primitive) idempotents.

For 1D irreps:

① $\underline{c} = \frac{1}{n!} \sum_{s \in S_n} s_n \qquad \underline{c}s = \underline{s}c = c \quad (\forall s \in S_n) \qquad \underline{c^2 = c}$

The subspace $\{\lambda c\}$ is an irrep.

$L(s) \cdot c = sc = c$

trivial irrep

② $\underline{c} = \frac{1}{n!}\sum_{s\in S_n} \mathrm{sgn}(s)\cdot s$

$$cs = sc = \mathrm{sgn}(s)\cdot c \quad \forall s\in S_n$$

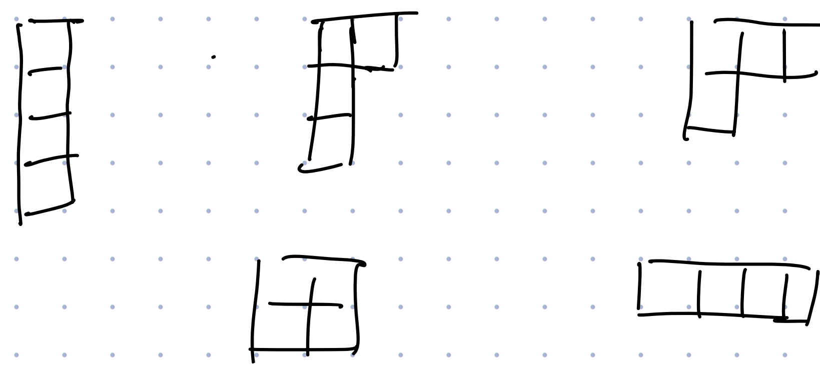
$$L(s)\cdot c = \mathrm{sgn}(s)\cdot c.$$

<u>sgn irrep</u>

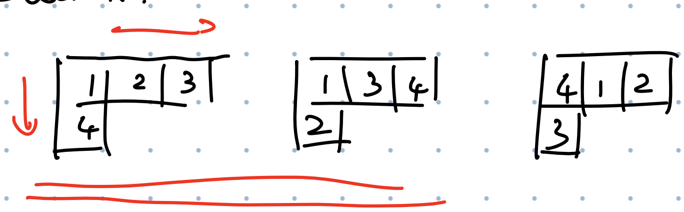
How to find projectors / idempotents onto other irreps?

⇒ use Young diagrams & Young tableaux.

<u>S_4</u>.



Young tableaux:



n! tableaux for a diagram

standard tableau: integers increase within row & column.

Given a tableau T. we define two sets of permutations $R(T)$. $C(T)$

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{aligned} R(T) &= \{e,\ (12),\ (13),\ (23),\ (123),\ (132)\} \\ C(T) &= \{e,\ (14)\} \end{aligned}$$

$$R(T) \cap C(T) = \{\underline{e}\}$$

$$\left(\begin{array}{l} p \in R(T),\quad q \in C(T) \qquad \underline{\underline{pq}} \text{ unique.} \\ p' \qquad\qquad\ \ q' \\ p'q' = pq \iff \underline{\underline{q(q')^{-1}}} = \underline{\underline{p^{-1} \cdot p'}} = e \implies p = p',\ q = q' \end{array} \right)^{(*)}$$

Then we construct two elements of $R_{S_n} =: R_n$

$$P = \sum_{p \in R(T)} p \qquad Q = \sum_{q \in C(T)} \epsilon(q) \cdot q \qquad \left(\begin{array}{l}\epsilon(q) = \mathrm{sgn}(q) \\ \in \{\pm 1\}\end{array} \right)$$

$$\underline{\underline{c}} = PQ = \sum_{\substack{p \in R(T) \\ q \in C(T)}} \epsilon(q)\, \underline{\underline{pq}} \quad \left(\overset{(*)}{\neq} 0 \right)$$

<u>Theorem 1</u>. $c = PQ$ corresponding to a tableau T is <u>essentially idempotent</u>

The invariant subspace $R_n c$ $(= \{gc,\ \forall g \in R_n\})$ yields an irrep of S_n.

That is to say:

① $c^2 = \lambda c$ ($\lambda > 0$ integers)

($\tilde{c} = \lambda^{-1} c$ idempotent)

$p^\mu p^\nu = p^\mu \delta_{\mu\nu}$

② $c \cdot c' = 0$ ($\forall c' \neq c$, T' a different tableau)

<u>Theorem 2.</u> The dimension f of the irrep corresponds to a diagram is the number of standard tableaux $\{T_i,\ i=1, \cdots f\}$

<u>Example</u>

S_4 $\begin{cases} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} & \text{trivial} \quad f = 1 \\ \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} & \text{sgn} \quad f = 1 \end{cases}$

S_3 $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$ $f = 2$ standard irrep.

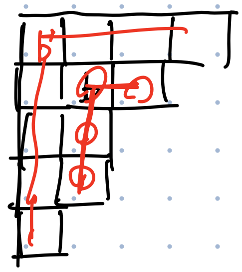
For a given T.

$$c(T)^2 = \lambda(T) c(T)$$

$$\lambda(T) = \frac{n!}{f} \qquad f: \text{dim of irrep.}$$

$$f = \frac{n!}{\prod_b h(b)} \qquad \text{"hook length formula"}$$

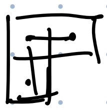
$h(b)$: hook length.



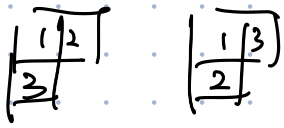
$h(b) = 4$

$h(b') = 8$

S_3.



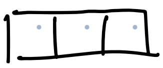
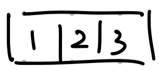
$f = \frac{3!}{3} = 2$

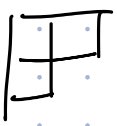
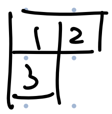
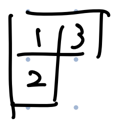


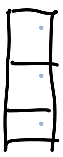
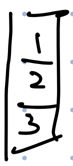
S_2 : $f = \frac{2!}{2} = 1$  $e - (12)$  $e + (12)$

Example S_3.

① diagrams — standard tableau(x)

trivial:  

standard.   

sgn :  

② trivial. $P = \sum_{p \in R(T)} p = e + (12) + (13) + (23) + (123) + (132)$

$Q = e$

$(\tilde{c}^2 = \tilde{c})$ $\quad \lambda = \frac{n!}{f} = 6$

$$\tilde{c} = \frac{1}{\lambda} c = \frac{1}{6}\left(e + (12) + (13) + (23) + (123) + (132) \right)$$

$\forall \phi \in S_3 \quad \phi \tilde{c} = \tilde{c}$

$R_{S_3} \cdot \tilde{c} = \tilde{c}$

sgn : $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$ $P = e$

$Q = e - (12) - (13) - (23) + (123) + (132)$

$\tilde{C} = \frac{1}{6} Q$

$\phi \tilde{C} = \mathrm{sgn}(\phi)\, \tilde{C} \quad (\phi \in S_3)$

$\{ R_{S_3} \cdot \tilde{C} \}$ 1D sgn

standard : $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$ $f = \frac{3!}{3} = 2$

T_1 T_2 $\lambda = 3$

T_1 : $P_1 = e + (12)$

$(12)(13) = (132)$

$Q_1 = e - (13)$

$$\begin{cases} \tilde{C}_1 = \frac{2}{6} P_1 \cdot Q_1 = \underline{\frac{1}{3}\left(e - (13) + (12) - (132) \right)} \\ \tilde{C}_2 = \frac{1}{3}\left(e - (12) + (13) - (123) \right) \end{cases}$$

$$\begin{cases} \tilde{C}_i \tilde{C}_i = \tilde{C}_i \\ \tilde{C}_1 \hat{C}_2 = 0 \end{cases}$$ check !

$\boxed{R_{S_3} \cdot \tilde{C}_1 :}$

$(12)(132) = (13)(2)$

$e \cdot \tilde{C}_1 = \tilde{C}_1 = \underline{\underline{v_1}}$

$\underline{(12) \cdot \tilde{C}_1} = \frac{1}{3}\left((12) - (132) + e - (13) \right)$

$= \underline{\tilde{C}_1}$

$(13)(132) = (1)(23)$

$$(13)\cdot\tilde{c}_1 = \frac{1}{3}\big((13) - e + (123) - (23)\big)$$

$$=: \underline{\underline{v_2}}$$

$$(23)\cdot\tilde{c}_1 = -v_1 - v_2$$

$$(123)\cdot\tilde{c}_1 = v_2$$

$$(132)\cdot\tilde{c}_1 = \underline{-v_1 - v_2}$$

Matrix rep. of $V = \operatorname{span}\{v_1, v_2\}$

$$\begin{cases} (12)\cdot v_1 = v_1 \\ (12)\cdot v_2 = (12)\big((13)\cdot v_1\big) = (132)\cdot v_1 = -v_1 - v_2 \end{cases}$$

$$M[(12)] = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad \chi_2((12)) = 0$$

$$\begin{cases} (13)\cdot v_1 = v_2 \\ (13)\, v_2 = v_1 \end{cases}$$

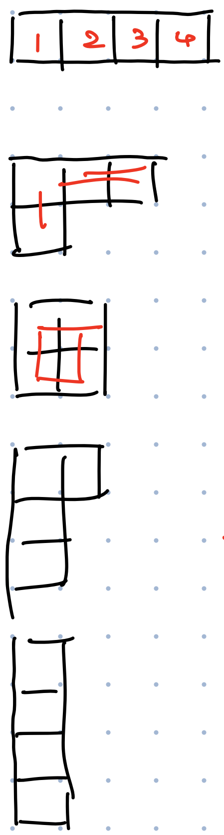
$$M[(13)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \underline{\chi_2((13)) = 0}$$

$$M[(23)] = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \chi_2((23)) = 0$$

$$M[(123)] = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \underline{\chi_2((123)) = -1}$$

Example: Character table of S_4.

1. Conjugacy classes ? 2. irreps ? $= \#$ conj. classes



(4) $f = \dfrac{4!}{\prod_b h(b)} = 1$ 1

(3)(1) $f = \dfrac{4!}{4 \times 2} = 3$

1 2 3 / 4   1 2 4 / 3   1 3 4 / 2   3

$(2)^2$ $f = \dfrac{4!}{3 \times 2 \times 2} = 2$

1 2 / 3 4   1 3 / 2 4   2

$(2)(1)^2$

1 2 / 3 / 4   1 3 / 2 / 4   1 4 / 2 / 3   3

$(1)^4$

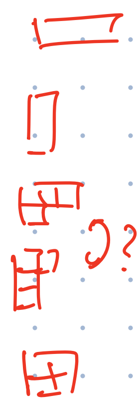
1 / 2 / 3 / 4   1

$|G| = \sum_\mu n_\mu^2$

$1 + 3^2 + 2^2 + 3^2 + 1 = 24 = 4!$

$\binom{4}{2} = 6$ $\binom{4}{2}/2$ $\binom{4}{3} \cdot 2$

	E	6 [(12)]	3 [(12)(34)]	8 [(123)]	6 [(1234)]
V^+	1	1	1	1	1
V^-	1	-1	1	1	-1
$V^\perp$	3	1	-1	0	-1
$V^- \otimes V^\perp$	3	-1	-1	0	1
V^2	2	0	2	-1	0



?

V^{R^n}	4	2	0	1	0

$S_n \quad \{e_i\} \quad \mathbb{R}^n \qquad L = \Sigma e_i$

$L^\perp =$

$$\underline{V^{\mathbb{R}^n}} \cong \underline{V^+ \oplus V^\perp}$$

$\langle \chi^\Gamma, \chi^\Gamma \rangle = 1 \iff$ irrep.