8.12. Orthogonality relations of characters ;

Charater table.

8.12.1 Orthogonality relations — (cont.)

isotypic decomposition of some rep V.

$$V \cong \oplus\, a_\mu V^{(\mu)}$$

$$\boxed{\begin{aligned} &\Rightarrow \chi_V = \sum_\mu a_\mu \chi_\mu \\ &a_\mu = \langle \chi_\mu , \chi_V \rangle = \int_G \overline{\chi_\mu(g)}\, \chi_V(g)\, dg \end{aligned}}$$

if $V \cong L^2(G)$ of a finite group.

$$\chi_V(e) = \dim V = |G|$$

$$\chi_V(g \neq e) = 0$$

$$a_\mu = \frac{1}{|G|} \sum_g \overline{\chi_\mu(g)}\, \chi_V(g) = \frac{1}{|G|} \big(\overbrace{n_\mu \cdot |G|}^{g=e} + \overbrace{0}^{g\neq e} \big) = n_\mu$$

$$|G| = \sum_\mu a_\mu \dim V^\mu = \sum_\mu n_\mu \cdot n_\mu = \sum_\mu n_\mu^2$$

Projection onto isotypic subspaces

$$P^\mu_{ij} := n_\mu \int_G \overline{M^{(\mu)}_{ij}(g)}\; T(g)\, dg$$

$$P^\mu_{ij} P^\nu_{kl} = \delta_{\mu\nu}\, \delta_{jk}\, P^\nu_{il}$$

$$T(h)\, P^\mu_{ij} = \sum_k M^\mu_{ki}(h)\, P^\mu_{kj}$$

Define $P^\mu := \sum_{i=1}^{n_\mu} P^\mu_{ii}$

$$P_\mu := \sum_{i=1}^{n_\mu} P^{(\mu)}_{ii} = \underline{n_\mu \int_G dg\, \overline{\chi_\mu(g)}\, T(g)}$$

$$P_\mu P_\nu = \sum_{i=1}^{n_\mu} \sum_{j=1}^{n_\nu} P^\mu_{ii} P^\nu_{jj} = \delta_{\mu\nu} \sum_{ij} \delta_{ij} P^\nu_{ij} = \delta_{\mu\nu} P_\nu$$

$$(P_\mu^2 = P_\mu)$$

$$P_\mu^\dagger = n_\mu \int_G \chi_\mu(g)\, T^\dagger(g)\, dg$$

unitary: $\chi_\mu(g) = \sum \lambda_i \quad |\lambda_i| = 1$

$\chi_\mu(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda}_i$

$$= n_\mu \int_G \chi_\mu^*(g^{-1})\, T(g^{-1})\, dg$$

$$= P_\mu$$

$\Rightarrow$ projectors onto isotypic subspaces

$\forall \psi \in V$: $\quad T(h)\, P^\mu \psi = T(h) \sum_{i=1}^{n_\mu} P^{(\mu)}_{ii}\, \psi = \sum_{ki} M^\mu_{ki}(h)\, P^{(\mu)}_{ki}\, \psi$

$$P^\mu \psi \in \mathcal{H}^\mu$$

$$\mathrm{Tr}(P^\mu) = \langle \psi, P^\mu \psi \rangle = n_\mu \underbrace{\int_G dg\, \overline{\chi_\mu(g)}\, \chi(g)}_{a_\mu} = n_\mu a_\mu$$

$$= \dim(\mathcal{H}^\mu \cong K^{a_\mu} \otimes V^{(\mu)})$$

8.12.2. Character table of finite groups

For finite groups,

we can define a set of class functions

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & \end{cases}$$

where C_i is a distinct conjugacy class.

$\{\delta_{C_i}\}$ is also a basis for the class functions $L^2(G)^{class}$

From above $\{\chi_\mu\}$ is a basis of $L^2(G)^{class}$

Theorem. The number of conjugacy classes of a finite group G = the number of irreps.

The character table is an $r \times r$ matrix

E $m_r = |C_r|$

		$m_1 C_1$	$m_2 C_2$	$\cdots$	$m_r C_r$
trivial Γ'	V^1	$\chi_1(C_1)$	$\chi_1(C_2)$	$\cdots$	
irreps $\rightarrow$	V^2	$\chi_2(C_1)$	$\chi_2(C_2)$	$\cdots$	
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	V^r				$\chi_r(C_r)$

$$\int_G dg\, \overline{\chi_\mu(g)}\, \chi_\nu(g) = \delta_{\mu\nu} \quad \Rightarrow$$

$$\frac{1}{|G|} \sum_{\{C_i\}} m_i\, \overline{\chi_\mu(C_i)}\, \chi_\nu(C_i) = \delta_{\mu\nu}$$

define $S_{\mu i} = \sqrt{\frac{m_i}{|G|}}\, \chi_\mu(C_i)$ then

$$\sum_{i=1}^{r} S_{\mu i} S^*_{\nu i} = \delta_{\mu\nu}.$$ S is a unitary matrix

$$S \cdot S^\dagger = \mathbb{1}_r$$

There is a dual orthogonality relation

$$\sum_\mu \overline{\chi_\mu(C_i)}\, \chi_\mu(C_j) = \frac{|G|}{m_i} \delta_{ij}$$

Examples

1. $S_2 \cong \mathbb{Z}_2$

	1	$[(12)]$
1^+	1	1
1^-	1	-1

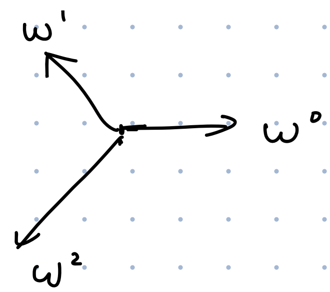
2. $G = \mathbb{Z}_n$ $\quad \# \{C_j\} = n$

$\# \{\text{irreps}\} = n$

$\mathbb{Z}_3$: $\quad \rho_m(\bar{j}) = (\omega_m)^j \qquad \omega_m = e^{i\frac{2\pi}{3}m}$

$= (\omega_1)^{mj} \qquad \omega = e^{i\frac{2\pi}{3}}$

	$[\bar{0}]$	$[\bar{1}]$	$[\bar{2}]$
ρ_0	1	1	1
ρ_1	1	ω	ω^2
ρ_2	1	ω^2	$\omega^{2\times 2} = \omega$



3. $G = S_3$

σ – 2 cycles $\qquad \tau$ — 3 cycles

$\sigma\tau\sigma = \tau^2 \qquad \tau\sigma\tau^{-1} = \sigma'$

	$[1]$	$3[(12)]$	$2[(123)]$
1^+	1	1	1
1^-	1	-1	1
2	2	A	B
		0	-1

Given a general rep & a character table. How do we find what irreps it reduces into ?

① $\mathbb{R}^3$ rep of S_3:

$$1 = \mathbb{1}_3 \qquad (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\chi_V = 3 \qquad\qquad 1 \qquad\qquad 0$$

$$a_\mu = \langle \chi_\mu , \chi_V \rangle = \int_G (\chi_\mu(g))^* \chi_V(g)\, dg$$

$$= \frac{1}{|G|} \sum_g \overline{\chi_\mu(g)}\, \chi_V(g)$$

	[1]	3[(12)]	2[(123)]
1^+	1	1	1
1^-	1	-1	1
2	2	0	-1
V	3	1	0

$$a_{1^+} = \frac{1}{6}(3 + 3\times 1 + 2\times 0) = 1$$

$$a_{1^-} = \frac{1}{6}(3 + 3\times(-1) + 2\times 0) = 0$$

$$a_2 = \frac{1}{6}((3\times 2) + 0 + 0) = 1$$

$$\chi_V = \chi_{1^+} + \chi_2$$

$$V \cong V_{1^+} \oplus V_2$$

② Regular rep of S_3. $\dim(L^2(S_3)) = |S_3| = 6$

$$\chi_V(e) = 6$$

$$\chi_V(g \neq e) = 0$$

$$a_\mu = \langle \chi_\mu , \chi_V \rangle = \frac{1}{|G|} \cdot |G| \cdot \chi_\mu(e) = \underline{\dim V^\mu}$$

$$\boxed{L^2(G) \cong \bigoplus_\mu (\dim V^\mu) \cdot V^\mu}$$

4. V a vector space. S_2 permutes on $V\otimes V$.

$$\sigma: \quad v_i\otimes v_j \longmapsto v_j\otimes v_i$$

$$\chi_{V\otimes V}(1) = d^2$$

$$\chi_{V\otimes V}(\sigma) = d \qquad (\text{only } i=j)$$

	1	σ
1^+	1	1
1^-	1	-1

$$a_{1^+} = \langle \chi^{1^+}, \chi_{V\otimes V}\rangle = \tfrac{1}{2}d(d+1)$$

$$a_{1^-} = \langle \chi^{1^-}, \chi_{V\otimes V}\rangle = \tfrac{1}{2}d(d-1)$$

$$V\otimes V = \tfrac{1}{2}d(d+1)\cdot V^{1^+} \oplus \tfrac{1}{2}d(d-1)V^{1^-}$$

$T_{ij}\, v_i\otimes v_j \in V\otimes V$. basis for

symmetric tensors. $\frac{1}{2}(e_i\otimes e_j + e_j\otimes e_i)$

antisymmetric tensors: $\frac{1}{2}(e_i\otimes e_j - e_j\otimes e_i)$

8.13 Decomposition of tensor products of representations.

V carries space of dim n, basis $\{v_1, \dots v_n\}$

W m $\{w_1 \dots w_m\}$

$V \otimes W$. dim $n \cdot m$ basis $\{ v_i \otimes w_j \quad 1 \le i \le n.\ 1 \le j \le m \}$

$$\sum_i a_i v_i \otimes \sum_j b_j w_j = \sum_{ij} a_i b_j \, v_i \otimes w_j$$

G-action $\quad g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$

rep. $\quad (T_1 \otimes T_2)(g)(v \otimes w) := T_1(g) \cdot v \otimes T_2(g) w.$

mat. rep. $\quad (M_1 \otimes M_2)(g)_{ia,jb} = [M_1(g)]_{ij} [M_2(g)]_{ab}$

character $\quad \chi_{T_1 \otimes T_2} = \chi_{T_1} \cdot \chi_{T_2}$

① particle of spin j_1
j_2 $\Rightarrow V^{j_1} \otimes V^{j_2}$
$\cong \bigoplus a_{j_3} V^{j_3}$

② many-particle system. local Hilbert space

$\mathcal{H}_i$ spin 1/2 fermion $= \{ \phi, \uparrow, \downarrow, \uparrow\downarrow \}$

$$\mathcal{H} = \bigotimes_i \mathcal{H}_i \Rightarrow \bigoplus_i a_i \mathcal{H}^{\text{conf}_i}$$

N.S.

$G \otimes U(1) \otimes SU(2)$
space group

Let (V_1, T_1) and (V_2, T_2) be two representations with isotypic decompositions (over field k)

$$V_1 = \bigoplus a_\mu V^\mu \qquad V_2 = \bigoplus b_\nu V^\nu$$

$$V_1 \otimes V_2 = \bigoplus \underline{a_\mu b_\nu} \, \underline{\underline{V^\mu \otimes V^\nu}}$$

$$V^\mu \otimes V^\nu \cong \bigoplus_\lambda \underline{\underline{N_{\mu\nu}^\lambda}} V^\lambda \qquad \left(\bigoplus D_{\mu\nu}^\lambda \otimes V^\lambda \right)$$

$$\underline{\underline{\chi_\mu \cdot \chi_\nu}} = \sum_\lambda N_{\mu\nu}^\lambda \chi_\lambda$$

fusion coefficient

Clebsch-Gordan for $SU(2)$

$$N_{\mu\nu}^\lambda = \langle \chi_\lambda , \chi_\mu \cdot \chi_\nu \rangle$$

for Finite groups

$$N_{\mu\nu}^\lambda = \frac{1}{|G|} \sum_{g \in G} \underline{\underline{\chi_\mu(g) \chi_\nu(g)}} \overline{\chi_\lambda(g)}$$

$m_i = |C_i|$

$$= \frac{1}{|G|} \sum_{\{C_i\}} m_i \chi_\mu(C_i) \underline{\chi_\nu(C_i)} \overline{\chi_\lambda(C_i)}$$

$$N_{\mu\nu}^\lambda = N_{\nu\mu}^\lambda \qquad \left(V^\mu \otimes V^\nu \cong V^\nu \otimes V^\mu \right)$$

<u>Examples</u>. 1. ρ_m of $\mathbb{Z}_N$ $\qquad \rho_m(j) = \left(e^{i \frac{2\pi}{N} m} \right)^j$

$$\rho_m \otimes \rho_n \cong \rho_{m+n}$$

$$N_{mn}^\lambda = \frac{1}{N} \sum_d \underline{e^{i \frac{2\pi}{N} (m+n) d} \, e^{-i \frac{2\pi}{N} \cdot \lambda d}}$$

$$= \delta_{m+n, \lambda}$$

2. irreps of S_3.

$$V^{1^+} \otimes V^{\mu} \cong \oplus_{\lambda} N_{1^+,\mu}^{\lambda} V^{\lambda}$$

$$N_{1^+,\mu}^{\lambda} = \frac{1}{|G|} \sum m_i \, \underline{\chi_{\mu}}(C_i) \overline{\underline{\chi_{\lambda}}(C_i)}$$

$$= \delta_{\mu\lambda}$$

$$\bigoplus_{\lambda} \delta_{\mu,\lambda} V^{\lambda} = V^{\mu}$$

$$\Rightarrow \underline{\underline{V^{1^+} \otimes V^{\mu}}} \cong V^{\mu}$$

<u>check</u>

$$V^{-} \otimes V^{-} \cong V^{+}$$

$$V^{-} \otimes V^{2} \cong V^{2}$$

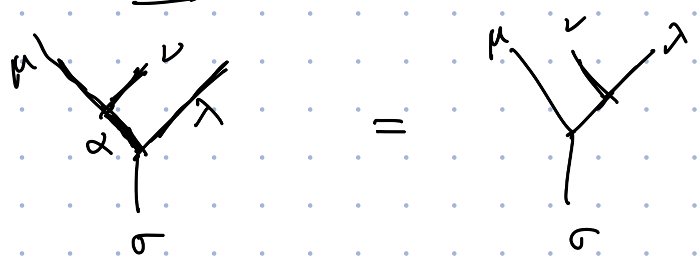
$$V^{2} \otimes V^{2} \cong V^{+} \oplus V^{-} \oplus V^{2}$$

$$\underline{(V^{\mu} \otimes V^{\nu}) \otimes V^{\lambda} \cong V^{\mu} \otimes (V^{\nu} \otimes V^{\lambda})}$$

$$\text{LHS} \cong \oplus_{\alpha} \underline{D_{\mu\nu}^{\alpha}} \, \underline{V^{\alpha}} \otimes \underline{V^{\lambda}}$$

$$\cong \oplus_{\sigma} (\oplus_{\alpha} \underline{D_{\mu\nu}^{\alpha} \otimes D_{\alpha\lambda}^{\sigma}}) \otimes V^{\sigma} \cong \oplus_{\sigma} (\oplus_{\beta} D_{\nu\lambda}^{\beta} \otimes D_{\mu\beta}^{\sigma}) \otimes V^{\sigma}$$

$$\sum_{\alpha} \underline{N_{\mu\nu}^{\alpha}} \, \underline{N_{\alpha\lambda}^{\sigma}} = \sum_{\beta} N_{\mu\beta}^{\sigma} N_{\nu\sigma}^{\beta}$$



"F-move"

digression : "Category theory"

TQFT / anyons / topo. quantum computation

$(X \otimes Y) \otimes (Z \otimes W)$ → pentagon relation

(ref. PRB 100. 115147)

Summary of key results

① unitary rep. of <u>compact</u> G.

$$\langle M^{\mu_1}_{i_1 j_1}, M^{\mu_2}_{i_2 j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$$

complete, orthogonal basis of $L^2(G)$.

② (Peter-Weyl) $L^2(G) \cong \bigoplus_\mu \mathrm{End}(V^\mu)$

$$\iota : \bigoplus_\mu \mathrm{End}(V^\mu) \to L^2(G)$$

$$\underline{\underline{\bigoplus_\mu S_\mu}} \mapsto \sum_\mu \varphi_{S_\mu}$$

$$\underline{\varphi_{S_\mu} := \mathrm{Tr}_{V_\mu}(S\, T(g^{-1}))}$$

↪ finite G: $\boxed{|G| = \sum_\mu n_\mu^2}$

$(n_\mu = \dim V^\mu)$

③ characters.

$$\int_G \overline{\chi^{(\mu)}(g)}\, \chi^{(\nu)}(g)\, dg = \delta_{\mu\nu}$$

ON basis of $L^2(G)^{\text{class}}$.

④ $V \cong \bigoplus_\mu a_\mu V^{(\mu)}$

$$a_\mu = \int_G \overline{\chi^{(\mu)}}(g)\, \chi_V(g)\, dg = \langle \chi^{(\mu)}, \chi_V \rangle$$

reg. rep. $a_\mu = \langle \chi^\mu, \underline{\chi} \rangle = \frac{1}{|G|} (\dim n_\mu) \cdot |G|$

$$\underline{= \dim n_\mu}$$

⑤ # irreps = # conj. class.

		C_i	- - - -
irreps	Γ_1		
	Γ_2		
	$\vdots$		

rows: $\frac{1}{|G|}\sum_{\{C_i\}} |C_i| \, \chi_\mu(C_i)\,\overline{\chi_\nu(C_i)} = \delta_{\mu\nu}$

columns: $\sum_{\mu} \overline{\chi_\mu(C_i)}\,\chi_\mu(C_j) = \frac{|G|}{m_i}\delta_{ij}$

8.13 Group algebra of finite groups.

Refs.

① Fulton & Harris. Representation theory (GTM 129)
Sec. 3.4.

☆ ② Miller. "Symmetry groups and their applications".
Chap 3.
Chap 4 symmetric group rep.

☆ ③ 陈金全. 第二章 群表示基础
"群元既是算符，又是基矢".

representation : $G \longrightarrow GL(V)$

Introduce a new vector space R_G (group ring)

Ring : set with $+$ and $\times$

① $+$: commutative ; 0 identity ; $-a$ inverse

② $\times$: (monoid). associative, identity

③ $a \times (b+c) = a \times b + a \times c$. distributive

§. 13.1. group algebra

Let G be a finite group of order n.

Define n-dim vector space R_G with basis

$\{ g . g \in G \}$

$$x = \sum_{g \in G} x(g) \cdot g \qquad x \in R_G \qquad x(g) \in \mathbb{C}.$$

$x = y$ iff $\forall g \in G$. $x(g) = y(g)$

$$x + y = \sum_{g \in G} x(g) \cdot g + \sum_{g \in G} y(g) \cdot g = \sum (x(g) + y(g)) g$$

$$\alpha x = \sum_{g \in G} \alpha x(g) \cdot g \qquad \alpha \in \mathbb{C}.$$

$$0 = \sum_{g \in G} 0 \cdot g$$

$$xy = \left(\sum_{g \in G} x(g) \cdot g \right) \left(\sum_{h \in G} y(h) h \right) = \sum_{g, h} x(g) y(h) gh$$

$$\overset{gh \to k}{=} \sum_k \left(\sum_g x(g) y(g^{-1} k) \right) \cdot k \equiv \sum_k xy(k) \cdot k$$

$$xy(k) = \sum_g x(g) y(g^{-1} k) \quad \text{convolution product}$$

$$\left((f * g)(t) = \int f(\tau) g(t - \tau) d\tau \right)$$

$\Rightarrow R_G$ is a group ring / group algebra $K[G]$

$\mathbb{C}[G]$

$x(g) \in \mathbb{Z}$ $\qquad$ $x(g) \in \mathbb{C}$