§10. Orthogonality relations of matrix elements of reps; Peter-Weyl theorem.

Recall: ① Basics of reg. rep.

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \int_G |f(g)|^2 dg < \infty \}$$

is a unitary $G \times G$

② V a rep. $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ is also a unitary rep of $G \times G$.

$S \in \mathrm{End}(V)$: $\underline{(g_1, g_2) \cdot S = T(g_1) \cdot S \cdot T(g_2)^{-1}}$

$$\iota : \mathrm{End}(V) \longrightarrow L^2(G)$$

$$S \longmapsto \underline{\mathrm{Tr}_V(S\, T(g^{-1}))} := \varphi_S$$

matrix unit $\underline{e_{ij}} \longmapsto M_{ij}^{Tr, -1} = M(g^{-1})_{ji}$

by a simple extension:

$$\iota : \underline{\bigoplus_\mu \mathrm{End}(V^\mu)} \longrightarrow \underline{L^2(G)}$$

$$\bigoplus_i S_i \longmapsto \sum_i \varphi_{S_i}$$

Peter–Weyl theorem: G compact. Then there is an isomorphism of $G\times G$ representations

$$L^2(G) \cong \oplus_\mu \mathrm{End}\,(V^{(\mu)})$$

where we sum over the distinct isomorphism class of each irrep exactly once.

Peter-Weyl theorem is the consequence of two statements.

1. Let (V, T) be a unitary irrep of a compact group G on a complex vector space V.

   Then V is finite dimensional.

Proof: pick a nonzero $v \in V$. Define, for $w \in V$.

$$L(w) = \int_G dg \langle T(g) v, w\rangle T(g) v$$

L is an operator $V \to V$

$$\begin{aligned} L(T(h) w) &= \int_G dg \langle T(g) v, T(h) w\rangle T(g) v \\ &= \int_G dg \langle T(h^{-1} g) v, w\rangle T(g) v \\ &\overset{h^{-1}g \to g}{=} \int_G dg \langle T(g) v, w\rangle T(hg) v \\ &= T(h) \int_G dg \langle T(g) v, w\rangle T(g) v \\ &= T(h) \cdot L(w) \end{aligned}$$

L is an intertwiner $\therefore$ $L\,T(h) = T(h)\,L \quad \forall h \in G$.

Schur's lemma $\Rightarrow$ $L = \lambda \mathbb{1}_V$. $\lambda \in \mathbb{C}$

$\langle v \cdot L(v) \rangle = \int_G dg\, |\langle T(g) v, v\rangle|^2$

$\hookrightarrow \equiv \lambda \|v\|^2$ $\qquad \underline{\lambda = \frac{1}{\|v\|^2} \int dg \langle T(g) v, v\rangle}$

$$\mathrm{Tr}(L) = \sum_i \langle v_i, L(v_i)\rangle$$
$$= \sum_i \int_G dg \langle T(g) v, v_i\rangle \langle v_i, T(g) v\rangle$$
$$= \sum_i \int_G dg\, |\langle v_i, T(g) v\rangle|^2$$
$$= \int_G dg\, \|T(g) v\|^2$$

$\|v\|^2$ due to unitary

$$= \|v\|^2 \, \mathrm{vol}(G) < \infty$$

$$\lambda \cdot \dim V = \|v\|^2 \mathrm{vol}(G)$$

$$\hookrightarrow \boxed{\dim V = \mathrm{vol}(G) \frac{\|v\|^4}{\int_G |\langle v, T(g) v\rangle|^2 dg}}$$

2. Let G be a compact group. The Hermitian inner product on $L^2(G)$

$$\langle \varphi_1, \varphi_2 \rangle := \underline{\int_G \varphi_1^*(g) \varphi_2(g)\, dg}$$

with normalized Haar measure. s.t. the volume of G $\int_G dg = 1$.

$L^2(G) \cong \oplus\, a^\mu V^{(\mu)}$

Let $\{V^\mu\}$ be a set of representations of distinct isomorphism classes of unitary irreps.

(Because of statement 1). For each $V^{(\mu)}$

choose an orthonormal (ON) basis $w_i^{(\mu)}$.

$i = 1, \ldots, n_\mu$. $\quad n_\mu = \dim V^{(\mu)}$

$$T^{(\mu)}(g)\, w_i^{(\mu)} = \sum_{j=1}^{n_\mu} M^{\mu}_{ji}(g)\, w_j^{(\mu)}$$

M^{μ}_{ij} form a complete orthogonal set of functions on $L^2(G)$.

$$\langle M^{\mu_1}_{i_1, j_1}, M^{\mu_2}_{i_2, j_2} \rangle = \frac{1}{n_\mu} \delta^{\mu_1 \mu_2} \delta_{i_1 i_2} \delta_{j_1 j_2}$$

Proof. $\forall A: V^\mu \to V^\nu$ a liner transf.

$$\tilde{A} := \int_G T^\nu(g) A T^\mu(g^{-1})\, dg$$

$$T^\nu(h)\tilde{A} = \int_G T^\nu(hg) A T^\mu(g^{-1})\, dg$$

$$\overset{g \to h^{-1}g}{=} \int_G T^\nu(g) A T^\mu((h^{-1}g)^{-1})\, dg$$

$$= \left(\int_G T^\nu(g) A T^\mu(g)^{-1}\, dg\right) T^\mu(h)$$

$$= \tilde{A}\, T^\mu(h)$$

$\tilde{A}$ is an intertwiner

$$\begin{array}{ccc} V^\mu & \xrightarrow{\tilde{A}} & V^\nu \\ \downarrow T^\mu & & \downarrow T^\nu \\ V^\mu & \xrightarrow{\tilde{A}} & V^\nu \end{array}$$

By Schur's lemma. $\tilde{A} = \delta_{\mu\nu} \hat{A}$. $\quad \underline{\hat{A} = c_A \mathbb{1}_\nu}$

Assign a basis for V^μ and V^ν

$$[\tilde{A}]_{ia} = \underline{\delta_{\mu\nu} C_A \cdot \delta_{ia}} = \int_G dg\, [M^\nu(g) A M^\mu(g^{-1})]_{ia}$$

$$= \underline{\sum_{i',a'} \int_G dg\, M^\nu_{ii'}(g) A_{i'a'} M^\mu_{a'a}(g^{-1})} \quad (*)$$

set $\mu = \nu$, $i = a$. and take the trace.

$$n C_A = \sum_{i,i',a'} \int_G dg\, M^\mu_{ii'}(g) A_{i'a'} M^\mu_{a'i}(g^{-1})$$

$$= \int_G dg\, \mathrm{Tr}\left(\cancel{M^\mu(g)}\, A\, \cancel{M^\mu(g^{-1})}\right)$$

$$= \int_G dg\, (\mathrm{Tr}\, A) = \mathrm{Tr}\, A$$

$$\Rightarrow \underline{C_A = \frac{1}{n_\mu} \mathrm{Tr} A}$$

Now take A to be the matrix unit e_{jk}

($\mathrm{Tr}\, e_{jk} = \delta_{jk}$).

insert into $(*)$

$$\sum_{i'a'} \int_G dg\, \underline{M^\nu_{ii'}}\overset{(g)}{} \overset{[e_{jk}]_{i'a'}}{(\delta_{j\underline{i'}}\, \delta_{k\underline{a'}})}\, M^\mu_{\underline{a'}a}(g^{-1}) = \overset{\frac{\delta_{jk}}{n_\mu}}{\frac{\mathrm{Tr}\, e_{jk}}{n_\mu}}\, \delta_{\mu\nu}\delta_{ia}$$

$$\Rightarrow \int_G dg\, \underline{M^\nu_{ij}(g)}\, \underline{M^\mu_{ka}(g^{-1})} = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ia}\delta_{jk}$$

$$\Downarrow$$

$$[M^\mu(g)^\dagger]_{ka} = \underline{\overline{M^\mu_{ak}(g)}}$$

$$\Rightarrow \langle M^\mu_{ak}, M^\nu_{ij} \rangle = \frac{1}{n_\mu}\delta_{\mu\nu}\delta_{ia}\delta_{jk}$$

$$\Rightarrow \langle M^{\mu_1}_{i_1,j_1}, M^{\mu_2}_{i_2,j_2} \rangle = \frac{1}{n_\mu}\delta^{\mu_1\mu_2}\delta_{i_1,i_2}\delta_{j_1,j_2}$$

We have shown that $\{ M^{\mu}_{ij} \}$ is a set of orthogonal functions on $L^2(G)$.

basis $\Leftarrow$ completeness ?

Let $\underline{W}$ be the subspace spanned by $\{ M^{\mu}_{ij} \}$.

$\Rightarrow$ The orthogonal complement $\underline{W^{\perp}}$ is also a unitary rep. of $G \times G$.

$\Rightarrow$ ecomposable into unitary irreps V^{μ}

$\{ f_j \}_{j=1}^{n_\mu}$ transforms as V^{μ} under right regular rep.

$$R(g) f_j = \Sigma M(g)^{\mu}_{kj} f_k$$

$$f(h\underline{g}) = \Sigma M(g)^{\mu}_{kj} f_k(h)$$

$h=1$
$$\Rightarrow f(g) = \sum_k f_k(1) \underline{M_{kg}^{\mu}(g)} \quad (\forall g \in G)$$

$f \in W$ contradiction with the assumption $f \in W^{\perp}$

$\Rightarrow W^{\perp} = 0$

[if with left reg. rep.

$$L(g) f_j = \sum \mu^\mu(g)_{kj} f_k$$

$$f(g^{-1}h) = \sum \mu^\mu(g)_{kj} f_k(h)$$

$$h=1 \Rightarrow f(g) = \sum \mu^\mu(g^{-1})_{kj} f_k(1)$$

$$= \sum \overline{\mu^\mu}(g)_{jk} f_k(1)$$

$\{\overline{\mu^\mu}_{ij}\}$ is another set of orthogonal basis]

$\Rightarrow \{\mu^\mu_{ij}\}$ is complete.

So now we know $\bigoplus_\mu \mathrm{End}\, V^\mu \cong L^2(G)$

Corollary for finite groups.

$L^2(G)$ of dim $|G|$:

$$\delta_a(g) = \begin{cases} 1 & g=a \\ 0 \end{cases}$$

$$(g\delta_a = \delta_{ga})$$

$\forall f: G \to \mathbb{C}$ $\quad f = \sum_{g_0 \in G} f(g_0)\delta_{g_0}$

$\mathrm{End}(V^\mu) \cong \mathrm{Mat}_{n_\mu \times n_\mu}(\mathbb{C})$ $\quad e_{ij}$

$\dim_{\mathbb{C}}(\mathrm{End}(V^\mu)) = n_\mu^2$

$$\Rightarrow |G| = \sum_\mu n_\mu^2$$

Examples 1. S_3 $\quad |S_3| = 6$

$6 = 1 \times 6 \quad \times$ abelian

$= 1 + 1 + 2^2 \quad ✓$

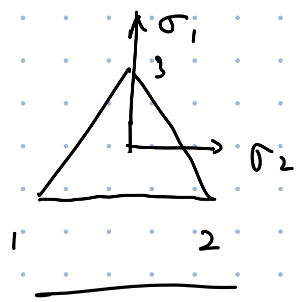
$$L(S_3) \underset{=}{\overset{u}{}} \underset{(+)}{\Gamma_{trivial}} \oplus \underset{(-)}{\Gamma_{sgn}} \oplus \underset{(2)}{2\Gamma_2}$$

① $M^+(\phi) = 1 \quad \forall \phi \in S_3$

② $M^-(\phi) = 1 \quad \phi \in \{(), (123), (132)\} = A_3$

$M^-(\phi) = -1 \quad \phi \in \{(12), (13), (23)\}$

③



$$\underline{M^{(2)}(12)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^{(2)}(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$M^{(2)}((23)) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\langle M^{\mu}_{ij}, M^{\mu'}_{i'j'} \rangle = \frac{1}{n_\mu} \delta_{\mu\mu'} \delta_{ii'} \delta_{jj'}$$

a. $\langle M^+, M^- \rangle = 0$

b. $\langle M^+, M^{(2)}_{11} \rangle = \frac{1}{6} \sum M^{(2)}_{11}(g) = \frac{2}{6}(1 - \frac{1}{2} - \frac{1}{2}) = 0$

c. $\langle M^{(2)}_{11}, M^{(2)}_{11} \rangle = \frac{2}{6}(1 + \frac{1}{4} + \frac{1}{4}) = \frac{1}{2} = \frac{1}{n_\mu}$

2. $G = \mathbb{Z}_2 = \langle \sigma \mid \sigma^2 = 1 \rangle$

$$\varphi \in L^2(G) = \{ Map(G, \mathbb{C}) \}$$

$$\varphi(1) = \varphi_+ \in \mathbb{C}$$

$$\varphi(\sigma) = \varphi_- \qquad L^2(G) \cong \mathbb{C}^2$$

$\mathbb{Z}_2$ irreps $\rho_\pm(\sigma) = \pm 1 \qquad V_\pm \cong \mathbb{C}$

$$\begin{cases} \mu^+(1) = \mu^-(1) = 1 \\ \mu^+(\sigma) = 1 \quad \mu^-(\sigma) = -1 \end{cases}$$

$\{\mu^+, \mu^-\}$ ON basis of $L^2(\mathbb{Z}_2)$

$$\Rightarrow \varphi = \frac{\varphi_+ + \varphi_-}{2} \mu^+ + \frac{\varphi_+ - \varphi_-}{2} \mu^-$$

Previously : $T(\sigma)$

$$\left(\begin{aligned} P_\pm &= \tfrac{1}{2}(\mu^\pm(1)\, \mathbb{1} + \mu^\pm(\sigma)\, T(\sigma)) \\ &= \tfrac{1}{2}(\mathbb{1} \pm T(\sigma)) \end{aligned} \right) \quad P_\pm = \tfrac{1}{2}(1 \pm T(\sigma)) \text{ is of the}$$

is form : $P_\pm = \int_G \overline{\mu^\pm(g)}\, T(g)\, dg$ (later)

3. $G = U(1) \qquad (\hat{G} = \mathbb{Z})$

(ρ_n, V_n): $\rho_n(z) = z^n \quad n \in \mathbb{Z} \quad (= \underline{e^{i\theta n}})$

$$V_n \cong \mathbb{C} \qquad \theta \in [0, 2\pi)$$

$$\langle \rho_{n_1}, \rho_{n_2} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} (\overline{\rho_{n_1}(\theta)})^* \rho_{n_2}(\theta) = \delta_{n_1, n_2}$$

$$e^{i\theta(n_1 - n_2)}$$

$\{\rho_n = e^{i\theta n}\}$ ON basis : $\Psi = \sum_n \alpha_n \rho_n$

$$\alpha_n = \int_{U(1)} \rho_n^* \Psi(g)\, dg$$

4. S_4? $\quad |S_4| = 24 = \underbrace{\overset{}{1}_{\text{trivial}} + \overset{\text{sgn}}{1} + \overset{}{3^2}_{\text{standard}}}_{11} + 13$?

S_4	e (⊞⊞⊞⊞)	(12)	$(12)(34)$	(123)	(1234)
ρ_{triv}	1	1	1	1	1
ρ_{sgn}	1	-1	1	1	-1
ρ_{std}	3	1	-1	0	-1
$\rho_{sgn} \otimes \rho_{std}$	3	-1	-1	0	1
ρ_2	2				

is $13 = 2^2 + 3^2$, or 1×9 or else?

(Problem 15 from HW)

$$[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}, \qquad [G, G] \triangleleft G.$$

$\rho: G \to \mathbb{C}^*$

$\rho([g_1, g_2]) = \rho(g_1 g_2 g_1^{-1} g_2^{-1}) = \rho(e) = 1$ trivial

$g[G,G] \in G/[G,G] \Rightarrow \rho(g[G,G]) = \rho(g)$

distinct 1D rep of G = distinct rep of $G/[G,G]$

$G/[G,G]$ abelian $\Rightarrow$ all irreps 1D

\# characters = \# conj. classes

$= |G/[G,G]|$

$[G, G] = A_n \Rightarrow |G/[G,G]| = 2.$

$\Rightarrow$ two distinct 1D irreps

8.11 Explicit decomposition of a representation

Let (T, V) be any rep. of a compact group G. Define

$$\underline{P^{(\mu)}_{ij}} := n_\mu \int_G \overline{\mathcal{M}^{(\mu)}_{ij}(g)}\, T(g)\, dg$$

$\mathcal{M}^{\mu}_{ij}$ w.r.t unitary irreps with ON basis of $V^{(\mu)}$.

$$\boxed{P^{(\mu)}_{i\underline{j}} P^{(\nu)}_{\underline{k}l} = \delta^{\mu\nu}\delta_{jk} P^{(\nu)}_{il}}$$

$$T(h) P^{\mu}_{ij} = n_\mu T(h) \int_G dg\, \overline{\mathcal{M}^{(\mu)}_{ij}(g)}\, T(g)$$

$$= n_\mu \int_G dg\, \overline{\mathcal{M}^{(\mu)}_{ij}(g)}\, T(hg)$$

$$\overset{hg\to g}{=} n_\mu \int_G dg\, \underbrace{\overline{\mathcal{M}^{(\mu)}_{ij}(h^{-1}g)}}_{\mathcal{M}^{(\mu)}_{ki}(h)\overline{\mathcal{M}^{(\mu)}_{kj}(g)}}\, T(g)$$

$$= \sum_{k}^{n_\mu} \mathcal{M}^{\mu}_{ki}(h) P^{(\mu)}_{kj}$$

$$T(h) P^{\mu j}_{i} = \sum_k \mathcal{M}^{\mu}_{ki}(h) P^{\mu j}_{k}$$

$\forall \varphi \in V$. $(P^{\mu}_{ij}\varphi \neq 0)$. then

span $\{ \underline{P^{\mu}_{ij}\varphi}, \ i=1, \cdots n_\mu \}$ (fix $\mu . j$)

transforms as (T^{μ}, V^{μ})

8.12. Orthogonality relations of characters ; Charater table.

8.12.1 Orthogonality relations

Recall. a class function on G:

$$f : G \to \mathbb{C}.$$

$f(g) = f(hgh^{-1})$ $\forall g, h \in G$. They span a subspace $L^2(G)^{class} \subset L^2(G)$.

<u>Theorem</u> The characters $\{\chi_\mu\}$ is an orthonormal (ON) basis for the vector space of class functions $L^2(G)^{class}$.

<u>Proof</u>. $\int_G dg\, M^{(\mu)}_{ij}(g)^* M^{(\nu)}_{kl}(g) = \frac{1}{n_\mu} \delta_{\mu\nu} \delta_{ik} \underline{\delta_{jl}}$

set $i=j$. $k=l$ & sum over i,k

$$\Rightarrow \int_G dg\, M^{(\mu)}_{ii}(g)^* M^{(\nu)}_{kk}(g) = \frac{1}{\underline{n_\mu}} \delta_{\mu\nu} \underline{\delta_{ik}}$$

$$\xRightarrow{\sum_{i,k}} \int_G dg\, \chi^\mu(g)^* \chi^\nu(g) = \delta_{\mu\nu}$$

$\Rightarrow \{\chi_\mu\}$ ON set

Completeness ?

$$\forall f \in L^2(G) \xrightarrow[\{M^\mu_{ij}\}\text{ complete}]{\text{Peter-Weyl}} f(g) = \sum_{\mu,i,j} \hat{f}^\mu_{ij} M^\mu_{ij}(g)$$

of $f \in L^2(G)^{\text{class}}$ $\quad f(g) = f(hgh^{-1})$

$$\underline{\int_G dh\, f(g)} = \int_G dh\, f(hgh^{-1})$$

$$\shortparallel \; f(g)$$

$$\int_G f(hgh^{-1})\,dh = \sum_{\mu,i,j} \hat{f}^\mu_{ij} \int_G \underbrace{M^\mu_{ij}(hgh^{-1})}_{M^\mu_{ik}(h) M^\mu_{kl}(g) M^\mu_{lj}(h^{-1})} dh$$

$$= \sum_{\substack{\mu,i,j\\k,l}} \hat{f}^\mu_{ij} M^\mu_{kl}(g) \underbrace{\int_G M^\mu_{ik}(h) M^{\mu *}_{jl}(h)\, dh}_{\frac{1}{n_\mu}\delta_{ij}\delta_{kl}}$$

$$= \sum_{\mu,i} \frac{\hat{f}^\mu_{ii}}{n_\mu} \chi_\mu(g)$$

$$\Rightarrow f(g) = \sum_{\mu,i} \frac{\hat{f}^\mu_{ii}}{n_\mu} \chi_\mu(g)$$

$\Rightarrow$ $\{\chi_\mu\}$ spans full $L^2(G)^{\text{class}}$.