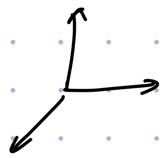
canonical rep of S_n on $\mathbb{R}^n$: $\hat{e}_i = (0, \cdots \overset{i\text{th}}{1}, \cdots 0)$

$\sigma \hat{e}_i = \hat{e}_{\sigma(i)}$



$L = \text{span}\{ \Sigma e_i \}$ invariant space.

$L^\perp = \{ \Sigma x_i e_i \mid \Sigma x_i = 0, \ x_i \in \mathbb{R} \}$

$\sum_{ij} x_i \langle e_i, e_j \rangle = \sum_{ij} x_i \delta_{ij} = \sum_j x_j = 0$

S_3: $L^\perp = \text{span}\{ e_1 - e_2, e_2 - e_3 \}$

We saw it is an irrep on $\mathbb{R}^2$ in lecture

is $L^\perp$ irrep in general ?

consider $u = \Sigma x_i e_i \in U$. $U \subset L^\perp$ an invariant subspace

not all x_i equal. otherwise $\Sigma x_i = 0 \Rightarrow x_i = 0$

WLOG. assumes $x_1 \neq x_2$

$u - \sigma_{12} u = x_1 e_1 + x_2 e_2 - x_1 e_2 - x_2 e_1$

$= (x_1 - x_2)(e_1 - e_2) \in U$

$\Rightarrow e_1 - e_2 \in U$

$\Rightarrow$ All $\tau \in S_n$ acts on $e_1 - e_2$

$(123)(e_1 - e_2) = e_2 - e_3 \in U$. etc.

$\Rightarrow \dim \text{span}\{ e_1 - e_2, e_2 - e_3, \cdots e_{n-1} - e_n \} = n-1$

$U = L^\perp$

7. Above examples are completely reducible.

Now consider example

a. $u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $x \in \mathbb{R}. \mathbb{C}.$

$\left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right\}$ is an invariant subspace.

b. $A \in GL(n, K)$

$$\det: \ A \mapsto \det A \mapsto \log|\det A|$$

$$A \cdot B \longmapsto \det AB = \det A \cdot \det B$$

$$\mapsto \log|\det A| + \log|\det B|$$

$$A \longrightarrow \begin{pmatrix} 1 & \log|\det A| \\ 0 & 1 \end{pmatrix}$$

$$T(A)\, T(B) = \begin{pmatrix} 1 & \log|\det A| + \log|\det B| \\ 0 & 1 \end{pmatrix} = T(AB)$$

8. Semidirect product $H \rtimes_\alpha G$

[recall direct product. $H \times G$

$$(h_1, g_1)(h_2, g_2) = (h_1 h_2, g_1 g_2)$$

semi-direct product has an additional

G-action of G on H:

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \alpha_{g_1}(h_2), g_1 g_2)$$

or. direct product is semidirect product with a trivial action. $\dashv$

$\{\alpha|\vec{\tau}\} \in$ Euclidean group. $\overset{\text{symmorphic}}{SG}$. $\alpha \in PG$.

$\vec{\tau} \in T$ translation

$$\{\alpha|\vec{\tau}\}\cdot\vec{r} = \cdot\vec{r} + \vec{\tau}$$

$$\{\alpha|\vec{\tau}_1\}\{\alpha_2|\vec{\tau}_2\}\vec{r} = \{R_1|\vec{\tau}_1\}(R_2\vec{r} + \vec{\tau}_2)$$
$$= R_1R_2\vec{r} + (R_1\vec{\tau}_2 + \vec{\tau}_1)$$
$$= \{R_1R_2 \,|\, R_1\vec{\tau}_2 + \vec{\tau}_1\}\vec{r}$$

the set $H \times G = \underset{\text{pure translation}}{T} \times PG \to$ point group.

$T \rtimes_R PG$: symmorphic space groups

$$(\vec{\tau}_1, \alpha_1)\cdot(\vec{\tau}_2, \alpha_2) = (\vec{\tau}_1 + R(\alpha_1)\vec{\tau}_2, \alpha_1\alpha_2)$$

matrix rep. $\begin{pmatrix} R & \tau \\ & 1 \end{pmatrix}$

<u>Proposition</u>: Let $(T.U)$ be a <u>unitary</u> rep. of on an inner product space V, and $W \subset V$ is an invariant subspace. The $W^\perp$ is an invariant subspace.

$$(W^\perp = \{y \in V \,|\, \langle y.x\rangle = 0 \;\forall x \in W\})$$

$\forall g \in G$, $y \in W^\perp$:

$$\langle T(g)y.x\rangle = \langle y, T(g)^\dagger x\rangle$$
$$= \langle y, T(g^{-1})x\rangle \underset{=}{\overset{T(g)x \in W}{}} 0$$

$\Rightarrow T(g)y \in W^\perp$, $\forall g \in G$

$\Rightarrow W^\perp$ invariant subspace.

Corollaries:

1. FD <u>uni.</u> rep. are always completely reducible

   If V reducible then $V = W \oplus W^{\perp}$

   if W reducible ?

   $W^{\perp}$ reducible ? $\rightarrow$ continue until irreducible

2. For compact groups, reps are unitarizable

   $\Rightarrow$ completely reducible.

3. Finite G. $L^2(G)$ is completely reducible

   [Recall that e.g. the Left rep on $L^2(G)$

   $$L_g \cdot \delta_h = \delta_{gh}. \qquad \delta\text{ - basis.}$$

   $|G|$ - dimensional rep.]

Example of reg. rep. of S_3

$$SMS^{-1} = \left(\begin{array}{c|c|c} O & & \\ \hline & O & \\ \hline & & O \end{array}\right)$$

$$\begin{cases} \chi(e) = |S_3| = 6 \\ \chi(g \neq e) = 0 \end{cases}$$

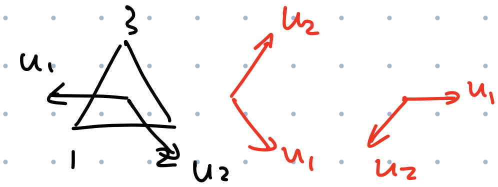
$\lambda_1 = 3$ $(l_1 = 3,\ l_2 = 0)$

conj. class

$\lambda_1 = 2$, $\lambda_2 = 1$, $l_1 = l_2 = 1$

character		$(\ )$	(123)	(12)
trivial	ρ_1	1	1	1
sgn	ρ_1'	1	1	-1
	ρ_2	2	-1	0

(123) (12)



$\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$ $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$tr = -1$ $tr = 0$

$$V^{ref} = x\,\rho_1 \oplus y\,\rho_1' \oplus z\,\rho_2$$

$$\begin{cases} x + y + 2z = 6 \\ x + y - z = 0 \\ x - y + 0\cdot z = 0 \end{cases} \Rightarrow \begin{cases} x = y = 1 \\ z = 2 \end{cases}$$

$$V^{reg.} = V^{\rho_1} \oplus V^{\rho_1'} \oplus 2V^{\rho_2}$$

- Isotypic components

Assume that the set of irreps (up to isomorphism) of G is countable

choose a representative $(T^{(\mu)}, V^{(\mu)})$, for each isomorphism class

$$V \cong \oplus_\mu \oplus_{i=1}^{a_\mu} V^{(\mu)}$$

a_μ is the number of times $V^{(\mu)}$ appears in the decomposition.

$\oplus_{i=1}^{a_\mu} V^{(\mu)}$ is the <u>isotypical component</u> of V associated to μ.

also, note that we can identify ($a_\mu \neq 0$)

$$\underbrace{V^{(\mu)} \oplus V^{(\mu)} \oplus \cdots \oplus V^{(\mu)}}_{a_\mu} \cong \underline{\underline{K^{a_\mu} \otimes V^{(\mu)}}} =: \underline{a_\mu} V^{(\mu)}$$

$$T(g) = \mathbb{1}_{a_\mu} \otimes T^{(\mu)}(g)$$

multiplicity/
degeneracy
space

Example 2. rep of $\mathbb{Z}_2$ on a vector space.

$$T : V \to V \qquad T \in \mathrm{Hom}(V, V)$$

$$\sigma^2 = 1 \Rightarrow T^2 = \mathbb{1}$$

$$P_\pm = \frac{1}{2}(1 \pm T) \qquad P_\pm^2 = \frac{1}{4}(1 + T^2 \pm 2T)$$

$$= \frac{1}{2}(1 \pm T) = P_\pm$$

$$\underline{V}^+ := \underline{\ker(P_+)} = \{ v \mid P_+ v = 0 \} = \underline{P_-(V)}$$

$$Tv = -v$$

$\underline{-1 \text{ eigenspace of } T}$

$$V^- = \ker(P_-) = \{ v \mid P_- v = 0 \} = P_+(V)$$

$$Tv = v$$

$\underline{+1 \text{ eigenspace}}$

$\mathbb{Z}_2$ has two 1D irreps. $\rho_+(1) = \rho_+(\sigma) = 1$

$\{1, \sigma\}$

$$\begin{cases} \rho_-(1) = 1 \\ \rho_-(\sigma) = -1 \end{cases}$$

$$V = K^{m+n} \text{ if } T(\sigma) = \mathrm{diag}(\underbrace{1, 1 \ldots 1}_{m}, \underbrace{-1, -1 \ldots -1}_{n})$$

$$V \cong \oplus_i^m e_+^i \oplus \oplus_i^n e_-^i$$

$$\cong K^m \otimes \rho_+ \oplus K^n \otimes \rho_- := m\rho_+ \oplus n\rho_-$$

8.8 Schur's lemmas

Lemma 1. Let G be any group. Let V_1, V_2 be vector spaces over any field k, s.t. they are carrier spaces of irreps of G.

If $A : V_1 \to V_2$ is an intertwiner between these two irreps, then A is either zero or an isomorphism of representations.
(invertible)

⌈ recall an intertwiner is a morphism of G-actions

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ {\scriptstyle T_1(g)}\downarrow & & \downarrow{\scriptstyle T_2(g)} \\ V_1 & \xrightarrow{A} & V_2 \end{array}$$

$$T_2(g)A = A\,T_1(g)$$ ⌋

Proof. $\ker A := \{ v_1 \in V_1 \mid A(v_1) = 0 \}$

$\operatorname{im} A := \{ v_2 \in V_2 \mid \exists\, v_1 \in V_1,\ \text{s.t. } v_2 = A(v_1) \}$

A an intertwiner, then

① $v_1 \in \ker A$. $A(T_1(g)\cdot v_1) = T_1(g)(Av_1) = 0$

$T_1(g)\, v_1 \in \ker A$, $\forall g \in G$

$\Rightarrow \ker A$ is an invariant subspace (of V_1)

② $v_2 \in \operatorname{im} A$ $\quad T_2(g)\, v_2 \overset{\exists v_1}{=} T_2(g) A \cdot v_1 = A(T_1(g)\cdot v_1) \in \operatorname{im} A$.

$\Rightarrow \operatorname{im} A$ inv. subspace. (of V_2)

V_1 is an irrep. $\Rightarrow$ ker A either 0 or V_1

if $\ker A = V_1$ then $A = 0$ (a)

else $\ker A = 0$, A is then injective. (b.1)

$(Av_1 = Av_2 \Rightarrow A(v_1 - v_2) = 0 \Rightarrow v_1 - v_2 \in \ker)$

which means im A cannot be $0 \Rightarrow \mathrm{im}\, A = V_2$

A is also surjective (b.2)

$\Rightarrow$ A is an isomorphism

skip

Now, set $V_1 = V_2 = V$. Then all $A: V \to V \in \mathrm{End}(V)$
$:= \mathrm{Hom}_G(V,V)$

form a endomorphism ring $(+, \times)$

$$\begin{cases} (A_1 \cdot A_2)v = A_1 \cdot (A_2 \cdot v) \\ (A_1 + A_2)v = A_1 v + A_2 v \end{cases}$$

Shur's lemma, A is invertible. a multiplication inverse is defined. $(AA^{-1} = 1)$

$\Rightarrow$ division ring / algebra. (non com. $\Rightarrow$ skew field, commutative $\Rightarrow$ field)

Examples: $\mathbb{R}$. $\mathbb{C}$. $\mathbb{H} \cong \mathrm{span}\{\mathbb{1}, i\sigma^k\}$
$\hookrightarrow$ quaternions

Lemma 2: Suppose (T, V) is an irrep of G
and V a finite-dim. complex vector space.
$A \in \mathrm{Hom}_G(V,V)$ $A: V \to V$ an $\mathbb{C}$-linear intertwiner. $(A\,T(g) = T(g)A, \forall g \in G)$

Then A is proportional to the identity transformation:

$$A(v) = \lambda v \quad (\lambda \in \mathbb{C})$$

Proof. $\exists v$. s.t. $Av = \lambda v$. i.e. there is always a nonzero eigenvector. it follows from the fact the $p(x) = \det(x\mathbb{1} - A)$ always has a root in $\mathbb{C}$. (fundamental theorem of algebra)

Then the eigenspace $C = \{w : Aw = \lambda w\}$ is nonzero.

$$A \underline{T(g) w} = T(g) A w = \lambda \underline{T(g) w} \quad \forall g \in G$$

$\Rightarrow$ C is an invariant subspace

$\overset{\text{irrep}}{\Rightarrow}$ $C = V$. $\Rightarrow A = \lambda \mathbb{1}$

Remarks.

1. If V_2 is completely reducible as $V_2 = W_1 \oplus W_2$

$$\mathrm{Hom}_G(V_1, W_1 \oplus W_2) \cong \mathrm{Hom}_G(V_1, W_1) \oplus \mathrm{Hom}_G(V_1, W_2)$$

$$\mathrm{Hom}_G(V^{(\mu)}, V) \cong \mathrm{Hom}_G\Big(V^{(\mu)}, \bigoplus_\nu K^{a_\nu} \otimes V^{(\nu)}\Big)$$

$$= \bigoplus_\nu K^{a_\nu} \otimes \underbrace{\mathrm{Hom}_G(V^{(\mu)}, V^{(\nu)})}_{\lambda \delta_{\mu\nu} \text{ by Schur's 1. lemma}}$$

$$= \underline{K^{a_\mu}} \otimes \underbrace{\mathrm{Hom}_G(V^{(\mu)}, V^{(\mu)})}_{\propto \mathbb{C}. \text{ 2.nd lemma}}$$

($\because$ $\mathrm{Hom}_G(V_1, V_2 \otimes V_3)$

$= V_2 \otimes \mathrm{Hom}_2(V_1, V_3)$

if G acts trivially on V_2)

K^{a_μ} is the linear space of G-invariant maps from $V^{(\mu)} \to V$. They can be thought as intertwiners.

There is a canonical equivariant map

$$\mathrm{Hom}_G(V^{(\mu)}, V) \otimes V^{(\mu)} \longrightarrow V$$
$$A \otimes v \longmapsto A(v) \in V.$$

and the isomorphism

$$\bigoplus \underbrace{\mathrm{Hom}_G(V^{(\mu)}, V)}_{\cong K^{a_\mu}} \otimes V^{(\mu)} \cong V$$

2. 1 is directly related to block diagonalization of Hamiltonians.

$\mathcal{H}$ the Hilbert space is a representation of some symmetry group G. and completely reducible

$$\mathcal{H} \cong \oplus_\mu \mathcal{H}^{(\mu)}$$

$$\mathcal{H}^{(\mu)} := D_\mu \otimes V^{(\mu)}$$

H is a Hamiltonian : $H : \mathcal{H} \to \mathcal{H}$. is an intertwiner. (commutes with G)

By schur's lemma,

$$H \overset{\cong}{=} \bigoplus_\mu \underbrace{H^{(\mu)}}_{} \otimes \overbrace{1_{V^{(\mu)}}}^{\text{Shur's lemma}}$$

Hermitian operators on D_μ

not determined by symmetry

We are familiar with this:

For H with certain symmetry. with a suitable basis transformation / choice

$$SHS^{-1} = \begin{pmatrix} H_{11} & 0 & 0 & \\ 0 & H_{22} & 0 & \\ 0 & 0 & H_{33} & \\ & & & \ddots \end{pmatrix}$$

Block diagonal. with blocks labeled by some "quantum number".

If an operator O, $[O, G] = 0$

$$O = \bigoplus_\mu O^{(\mu)} \times 1_{V^{(\mu)}}$$

$$\langle \varphi_1, O\varphi_2 \rangle = 0 \quad \text{if} \quad \varphi_1 \in \mathcal{H}^{(\mu)} \quad \varphi_2 \in \mathcal{H}^{(\nu)} \quad (\mu \neq \nu)$$

Example. $\mathbb{Z}_2$ action on $\mathcal{H}$.

$T^2 = 1$ $\qquad$ $|1\rangle$ $\quad$ $|2\rangle$

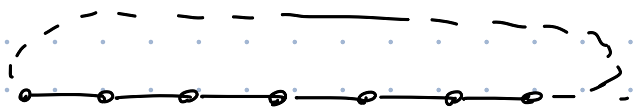
$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

$$= K^n \otimes \rho_+ \oplus K^n \otimes \rho_-$$

$$\rho_+ = \left\{ \tfrac{1}{\sqrt{2}} (|1\rangle + |2\rangle) \right\} \quad \rho_- = \left\{ \tfrac{1}{\sqrt{2}} (|1\rangle - |2\rangle) \right\}$$

generalize $T^2=1$ $\Rightarrow$ 1D tight-binding model

$$\hat{H} = -t \sum_{\langle ij \rangle} \overset{|i\rangle\langle j|}{a_i^\dagger a_j} + h.c.$$



$$G = C_N = \langle T \mid T^N = 1 \rangle$$

$$\tilde{T} = \mathrm{diag}\{e^{ik_1}, e^{ik_2}, \dots e^{ik_N}\}$$

$$\mathcal{H} = \oplus \mathcal{H}^\ell \qquad \mathcal{H}^\ell = \sum_j e^{ik_\ell \cdot j} |j\rangle$$

$$\tilde{T} \sum e^{ik_\ell \cdot j} |j\rangle = \sum_j e^{ik_\ell j} |j+1\rangle$$

$$= e^{-ik_\ell} \sum_j e^{ik_l (j+1)} |j+1\rangle$$

$$\langle k_\ell | H | k_m \rangle = -2t \cos k_\ell \, \delta_{\ell m}$$

in general

$$H_{(\mu_1, i_1, \alpha_1), (\mu_2, i_2, \alpha_2)} = \underbrace{\delta_{\mu_1 \mu_2}}_{\text{1st lemma}} \overbrace{\delta_{i_1 i_2}}^{\text{2nd lemma}} h_{\alpha_1 \alpha_2}$$

$$i_j = 1, \dots, n_\mu$$

8.9 Pontryagin duality <u>skipped</u>