8.7 Reducible & irreducible representations

Recall the direct sum of reps.

$$T_{V\oplus W} = T_V \oplus T_W$$

$$M_{V\oplus W} = \left(\begin{array}{c|c} M_V & 0 \\ \hline 0 & M_W \end{array}\right)$$

Quite often, instead, we would like to

"reduce" a representation of large dimension into representations of smaller dimensions.

<u>Definition</u> Let $W \subset V$ be a linear <u>subspace</u> of carrier space V of a group rep. $T: G \to GL(V)$. Then W is <u>invariant</u> under T, a.k.a an <u>invariant subspace</u> if $\forall g \in G$, $w \in W$.

$$T(g)\, w \in W.$$

Example

1. $\{\vec 0\}$ & V

2. $\mathbb{R}^3$ under $SO(2)_{\hat z}$: xy plane is a subspace

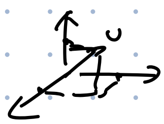
quiz here: (other planes at finite z_0 are not)

3. canonical rep. of S_n:

$$T(\phi): \vec e_i \to \vec e_{\phi(i)}$$

Then $\vec v = \vec e_1 + \vec e_2 + \dots + \vec e_n$ is invariant

$$T(\phi)\vec v = T(\phi)\sum_i \vec e_i = \sum_i \vec e_{\phi(i)} = \vec v$$

in $\mathbb{R}^3$:  diagonal vector

4. Mat rep.

$$M : G \to GL(n, K)$$

M_{ij} as a function: $G \to K$

$$g \mapsto M_{ij}(g)$$

The linear span of M_{ij} with <u>fixed i</u>

$$R_i := \mathrm{span}\{M_{ij},\ j = 1, \dots n\}$$

<u>right action</u>:

$$(\underbrace{R(g)\cdot M_{ij}}_{\substack{M' \\ \text{a function}}})(h) = M_{ij}(hg)$$

$$= \underbrace{\sum_s M_{sj}(g)}_{\text{coefficients}} M_{is}(h)$$

$\Rightarrow R_i$ is an invariant subspace

<u>left action</u> :

$$\mathcal{L}_j := \operatorname{span}\{ M_{ij} , i = 1, \dots n \}$$

is also invariant

$\Rightarrow \mathcal{LR} = \operatorname{span}\{ M_{ij} , \ i,j = 1, - n \}$ subspace of $L^2(G)$

is invariant under $G \times G$ -action

$$((g_1, g_2) \cdot f)(h) = f(g_1^{-1} h g_2)$$

note under left G action.

Remarks

$$\mathcal{LR} \cong \oplus_i^n \mathcal{L}_i$$

1. (T, V) a rep. $\exists W \subset V$ an invariant subspace. then we can restrict T to W.

$(T|_W, W)$ is a <u>subrepresentation</u> of (T, V)

$$T|_W(g) = T(g)|_W$$

We will write T instead of $T|_W$.

2. if T is unitary on V then it is unitary on W.

$$\langle T v_1, T v_2 \rangle = \langle v_1, v_2 \rangle \quad \forall v_i \in V.$$

Definition. A representation (T, V) is **reducible**

if there is a **proper**, **nontrivial** invariant subspace $W \subset V$ $(W \neq 0, V)$

If V is not reducible, it is an **irreducible representation** ("irrep")

Remarks.

1. $\forall v \in V$. $\text{span}\{T(g)v, \forall g \in G\}$ is an invariant subspace.

   If T is an irrep. it is V.

   such a vector is called a cyclic vector.

Note: the existence does not imply that the representation is irreducible

Consider e_1 in the **permutation representation**.

$\sum e_i$ is a proper, nontrivial invariant subspace

2. (T, W) a subrep of (T, V)

Choose an ordered basis

$$\{w_1, \cdots w_k\}$$

Then it can be completed to an ordered basis of V

$$\{w_1, \cdots w_k, u_{k+1}, \cdots u_n\}$$

$$T(g)(w_i) = (M_{11}(g))_{ji} w_j + (M_{21}(g))_{ai} u_a$$

$$T(g)(u_a) = (M_{12}(g))_{ja} w_j + (M_{22}(g))_{ba} u_b$$

i.e. $$(w, u)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

W invariant $\Rightarrow M_{21} = 0$

$$\Rightarrow T(g)(w_i) = \sum_j M_{11}(g)_{ji} w_j$$

$$\begin{pmatrix} M_{11}^{g_1} & M_{12}^{g_1} \\ 0 & M_{22}^{g_1} \end{pmatrix}\begin{pmatrix} M_{11}^{g_2} & M_{12}^{g_2} \\ 0 & M_{22}^{g_2} \end{pmatrix} = \begin{pmatrix} \boxed{M_{11}^{g_1} M_{11}^{g_2}} & M_{11}^{g_1} M_{12}^{g_2} + M_{12}^{g_1} M_{22}^{g_2} \\ 0 & M_{22}^{g_1} M_{22}^{g_2} \end{pmatrix}$$

M_{11} is a rep on W

M_{22} is not a rep on $V \setminus W$.

What if we want to further simplify it?

If we define a change of basis $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$

$$(w, u)\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = (w, wS + u) \equiv (w, u')$$

$$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -S \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} M_{11}(g) & M_{12}(g) \\ 0 & M_{22}(g) \end{pmatrix} = \begin{pmatrix} M_{11}(g) & M_{12}(g) - S M_{22}(g) \\ 0 & M_{22}(g) \end{pmatrix}$$

we require $M_{12}(g) - S M_{22}(g) = 0 \quad \forall g \in G$.

This puts a stronger restriction on the structure of the representation.

3. quotient space. V/W.

$$v_1 \sim v_2 \text{ iff } v_1 - v_2 \in W.$$

$$T(g)(v+W) = T(g)(v) + W$$

$$\Rightarrow T(g_1)T(g_2)(v+W) = T(g_1)(T(g_2)v + W)$$
$$= T(g_1)T(g_2)v + W$$
$$= [T(g_1)T(g_2)](v+W)$$

We define a basis for V/W as $e_a + W$. The rep looks like M_{22} wrt this basis.

<u>Definition</u> A representation T is called completely reducible if it is isomorphic to a direct sum of representations.

$$W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

where W_i are irreps. Thus there is a basis in which the matrices look like

$$M(g) = \begin{pmatrix} M_{11}(g) & 0 & 0 & \cdots \\ 0 & M_{22}(g) & \cdots & \\ 0 & \ddots & M_{33}(g) & \\ \vdots & & & \ddots \end{pmatrix}$$

$\boxed{\text{irreps are completely reducible.}}$

reducible but not completely $\Rightarrow$ "indecomposable"

Examples

1. $G = \mathbb{Z}_2$ 1-D rep $V = \mathbb{R}$

trivial : $\rho_+(1) = \rho_+(-1) = 1$

$\rho_-(1) = 1, \quad \rho_-(-1) = -1$

2. $G = \mathbb{Z}_2 \cong S_2 = \{e, \tau\}$

$$M(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow \tilde{M}(\tau) = A^{-1} M A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\rho_+(e) = \rho_+(\tau) = 1$

$\rho_-(e) = 1, \quad \rho_-(\tau) = -1$

$(T, V) \cong \rho_+ \oplus \rho_-$ completely reducible

3. $G = U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$ $V = \mathbb{C}$.

$\rho_n(z) = z^n$ for $\forall n \in \mathbb{Z}$.

$\rho_n(z_1 z_2) = (z_1 z_2)^n = \rho_n(z_1)\, \rho_n(z_2)$

<u>are there other irreps ?</u>

4. Finite-dimensional representations of <u>Abelian</u> groups are completely reducible.

Choosing an ordered orthonormal (ON) basis. s.t. all $M(g)$ ($\forall g \in G$) are commuting <u>unitary</u> matrices. over the complex field

$$M(g_i)M(g_j) = M(g_j)M(g_i) \quad \forall g_i, g_j \in G$$

as required by the abelianity.

$\Rightarrow$ M's can be simultaneously diagonized. (spectral theorem)

$$M(z) = \mathrm{diag}\{\lambda_1(z), \lambda_2(z), \cdots \lambda_d(z)\}$$

For $G = U(1)$. any f.d. rep on $V \cong \mathbb{C}^d$

$$M(z) = \mathrm{diag}\{\rho_{n_1}(z), \rho_{n_2}(z), \cdots \rho_{n_d}(z)\}$$

$$V \cong \rho_{n_1} \oplus \rho_{n_2} \oplus \cdots \oplus \rho_{n_d}.$$

Finite. compact Abelian groups all irreps are 1D.

e.g. $SO(2)$ $\quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

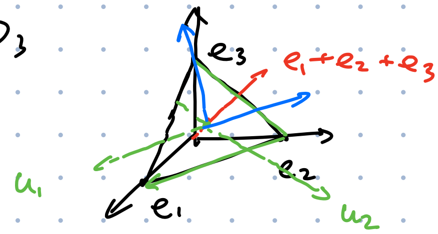
$$\rightarrow \left(\begin{array}{c|c} e^{i\theta} & 0 \\ \hline 0 & e^{-i\theta} \end{array}\right)$$

So reducible on $\mathbb{C}$. but irreducible on $\mathbb{R}$.

5. Non abelian $S_3 \cong D_3$

on $\mathbb{R}^3 = \mathrm{span}\{e_1, e_2, e_3\}$

$T(\sigma)e_i = e_{\sigma(i)}$



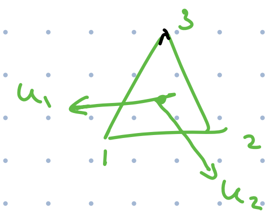
① $u_0 = e_1 + e_2 + e_3$ invariant subspace W

$T(\sigma)u_0 = u_0 \Rightarrow T|_W = \mathbb{1}_W$. trivial rep.

② its complement $W^\perp = \mathrm{span}\{u_1, u_2\}$

a. $u_1 = e_1 - e_2$

$u_2 = e_2 - e_3$



$T((12)) \cdot u_1 = -u_1$

$T((12)) \cdot u_2 = u_1 + u_2$

$M((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$T((23))\, u_1 = u_1 + u_2$

$T((23))\, u_2 = -u_2$

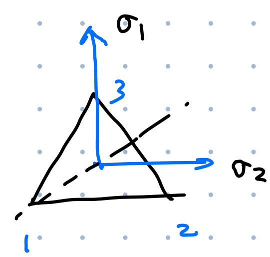
$M((23)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

$T((13))\, u_1 = -u_2$

$T((13))\, u_2 = -u_1$

$M((13)) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

unitary rep. not unitary mat.



b. using ON basis.

$M((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$T[(23)]\sigma_1 = -\frac{1}{2}\sigma_1 + \frac{\sqrt{3}}{2}\sigma_2$

$T[(23)]\sigma_2 = \frac{\sqrt{3}}{2}\sigma_1 + \frac{1}{2}\sigma_2$

$M[(23)] = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

similarly.

$M[(13)] = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ $\quad M[(123)] = R(\frac{2}{3}\pi) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$\mathbb{R}^3 \cong W \oplus W^\perp$

6. more generally. consider rep. of S_n on $\mathbb{R}_n$

$u_0 = \Sigma e_i$ invariant subspace W

$L = \{ x \Sigma e_i \ , x \in \mathbb{R} \}$

$L^{\perp} = \{ \Sigma x_i e_i \mid \Sigma x_i = 0 \ , x_i \in \mathbb{R} \}$

Both L and $L^{\perp}$ are irreducible.