Recap: representation theory

1. $$G \xrightarrow{\text{rep.}} GL(V) \xrightarrow[\cong]{\text{ordered basis } \{\hat{v}_i\}} GL(n,k)$$
$$g \longmapsto T(g) \qquad T(g)\hat{v}_i = \sum_j M(g)_{ji}\hat{v}_j$$

$V = k^n$ carrier space / rep. space

n: dim of rep.

2. intertwiner: equivariant linear map $V_1 \to V_2$

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "A"] & V_2 \end{tikzcd}$$

$A T_1(g) = T_2(g) A$

$A \in \mathrm{Hom}_G(V_1, V_2)$

if A invertible. $T_1(g)A^{-1} = A^{-1}T_2(g)$

$$\begin{tikzcd} V_1 \arrow[d, "T_1"'] & V_2 \arrow[l, "A^{-1}"'] \arrow[d, "T_2"] \\ V_1 & V_2 \arrow[l, "A^{-1}"] \end{tikzcd}$$

3. equivalent rep. $T_2(g) = A T_1(g) A^{-1}$

4. character as a class function $\chi_T(g) = \mathrm{Tr}_V(T(g))$

$\chi_T(hgh^{-1}) = \chi_T(g) \quad g, h \in G$

5. $T_1 \oplus T_2 \quad \chi_\oplus = \chi_1 + \chi_2 \qquad T_1 \otimes T_2 \quad \chi_\otimes = \chi_1 \cdot \chi_2$

6. unitary rep.

inner product space $\langle \alpha\varphi, \beta\phi \rangle = \bar{\alpha}\beta\langle \varphi, \phi \rangle$

$\langle U(g)v, U(g)v \rangle = \langle v, v \rangle \quad g \in G \quad v \in V$

↳ equivalent to uni. rep

unitarizable.

finite groups:

$$H = \sum_{g \in G} T^{\dagger}(g) T(g)$$

① $$\tilde{T}(g) = \Lambda^{\frac{1}{2}} V^{\dagger} T(g) V \Lambda^{-\frac{1}{2}}$$
$$\Lambda = V^{\dagger} H V$$

② new inner prod.: $\langle u, u \rangle_H = \frac{1}{|G|}\sum_{g} \langle T(g)u, T(g)u \rangle$

invariant. $\sum f(g) \xrightarrow{h} \sum f(hg)$

7. Summation / integration over G.

→ Haar measure

(invariant integration)

§.5 Haar measure (invariant integration)

$$f: G \to \mathbb{C} \qquad \text{left translation:}$$
$$g \mapsto f(g) \qquad (L_h^* f)(g) := f(hg)$$

$$\frac{1}{|G|}\sum_g f(g) = \frac{1}{|G|}\sum_g f(hg) = \frac{1}{|G|}\sum_g f(gh)$$

left invariance — right invariance

$$\int_G f(g)\, d\mu(g) = \int_G f(hg)\, d\mu(g) \qquad \text{left Haar measure } \mu(g)$$

1. $G = \mathbb{R}$

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \int_{\mathbb{R}} f(a+x)\, d\mu(x)$$

$$\Rightarrow d\mu(x) = c\, dx \qquad \mu(x) = \int dx$$

2. $G = \mathbb{Z}$ $\quad \int_G f(g)\, d\mu(g) = c \sum_{n\in\mathbb{Z}} f(n)$

$$\Rightarrow \mu(g) = c \sum_{n\in\mathbb{Z}}{}'$$

Consider more complex groups with multiplication

3. $G = \mathbb{R}^*_{>0}$ $\quad \int_G d\mu(g) f(g) = \int_G d\mu(ax) f(ax) \overset{\text{Haar}}{=} \int_G d\mu(x) f(ax)$

$$d\mu(x) = \frac{dx}{x}$$

$$\forall a \in \mathbb{R}^*_{>0}: \ \int_0^\infty f(ax)\frac{dx}{x} = \int_0^\infty f(x) \frac{d(x/a)}{x/a} = \int_0^\infty f(x)\frac{dx}{x}$$

4. $G = U(1) = \{ z \in \mathbb{C} . \ |z| = 1 \}$

$$\int_{U(1)} d\mu(z)\, f(z) = \int_{U(1)} d\mu(z)\, f(z_0 z)$$

$$g(\phi) = f(e^{i\phi})$$

$$= \int_{U(1)} d\mu(z_0^{-1} z)\, f(z) \qquad = \int_0^{2\pi} \frac{d\phi}{2\pi}\, g(\phi)$$

$$d\mu(z) = \frac{dz}{z} \qquad z = e^{i\phi} \quad dz = iz\, d\phi$$

$$\int_{U(1)} d\mu(z) f(z) = \frac{1}{2\pi i} \int_{U(1)} f(z) \frac{dz}{z} \qquad = \int_0^{2\pi} \frac{d\phi}{2\pi}\, g(\phi)$$

5. $G = GL(n, \mathbb{R})$ $\quad g \longmapsto g_0 g = g'$ $\quad \underline{\underline{g \in \mathbb{R}^{n^2}}}$

$$g'_{ij} = \sum_k (g_0)_{ik}\, g_{kj} \Rightarrow \frac{\partial g'_{ij}}{\partial g_{kl}} = (g_0)_{ik}\, \delta_{jl}$$

$$\frac{\partial g'_{ij}}{\partial g_{kj}} = (g_0)_{ik}$$

$$\prod_{ij} d g'_{ij} \longmapsto \underline{\left| \frac{\partial (g'_{11} \cdots g'_{nn})}{\partial (g_{11} \cdots g_{nn})} \right|} \prod_{ij} d g_{ij}$$

$$\begin{array}{c|l} & 11 \ 21, \cdots ;\ 12, 22, \cdots ;\ 13, 23 \cdots \\ \hline 11 & \\ 21 & \begin{pmatrix} g_{11} & g_{12} \\ & \end{pmatrix} \quad 0 \quad\quad 0 \\ \vdots & \quad 0 \quad \begin{pmatrix} & \end{pmatrix} \quad 0 \\ & \quad 0 \quad\quad 0 \quad \begin{pmatrix} & \end{pmatrix} \\ & \quad 0 \end{array}$$

$$\det \bigoplus_i M_i = \prod_i \det M_i$$

$$= |\det g_0|^n \prod_{ij} d g_{ij}$$

Haar measure $\quad \mu(g) := c \int |\det g|^{-n} \prod_{ij} d g_{ij}$

$$\int f(g_0 g)\, |\det g|^{-n} \prod d g_{ij}$$

$$= \int f(g)\, |\det g_0^{-1} g|^{-n} \prod \underline{d(g_0^{-1} g)_{ij}}$$

$$= \int f(g)\, |\det g|^{-n}\, \cancel{|\det g_0|^{n}}\, \cancel{|\det g_0|^{-n}} \prod d g_{ij}$$

$$= \int f(g)\, |\det g|^{-n} \prod d g_{ij}$$

6. $G = SU(2)$ $\quad g \in SU(2)$

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$$

$$u(\phi, \theta, \psi) = U_z(\phi) U_x(\theta) U_z(\psi)$$

$$= e^{i\frac{\sigma_z}{2}\phi} e^{i\frac{\sigma_x}{2}\theta} e^{i\frac{\sigma_z}{2}\psi}$$

$$= \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\psi}{2}} & \\ & e^{-i\frac{\psi}{2}} \end{pmatrix}$$

$$\begin{cases} \alpha = e^{\frac{i}{2}(\phi+\psi)} \cos\frac{\theta}{2} \\ \beta = i e^{\frac{i}{2}(\phi-\psi)} \sin\frac{\theta}{2} \end{cases}$$

$$(\phi, \psi) \longrightarrow \begin{cases} \phi + 4\pi, \ \psi \\ \phi, \ \psi + 4\pi \\ \phi + 2\pi, \ \psi + 2\pi \end{cases}$$

$$\begin{cases} \theta \in [0, \pi) \\ \phi \in [0, 2\pi) \\ \underline{\underline{\psi \in [0, 4\pi)}} \end{cases}$$

① $$d\alpha\, d\bar{\alpha}\, d\beta\, d\bar{\beta} \longrightarrow \underbrace{\left| \frac{\partial(\alpha, \bar{\alpha}, \beta, \bar{\beta})}{\partial(r, \phi, \theta, \psi)} \right|}_{J} dr\, d\psi\, d\phi\, d\theta$$

$$J = \frac{1}{2} r^3 \sin\theta \Big|_{r=1} = \frac{1}{2}\sin\theta$$

② $g \mapsto g_0 g$. $\quad |\det g_0| = 1 \quad (\because SU(2))$

note the parametrization ?

$$\mu(g) = \cancel{c}_{\frac{1}{16\pi^2}} \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} \underline{\underline{\sin\theta}}\, d\theta \int_0^{4\pi} d\psi}_{2\pi \times 2 \times 4\pi = 16\pi^2}$$

the form will be different

7. $SU(2) \xrightarrow{\pi} SO(3)$

$u\,\vec{x}\cdot\vec{\sigma}\,u^{-1} = (\pi(u)\vec{x})\cdot\vec{\sigma}$ $\pi(u) \in SO(3)$

euler angles
$$\left\{\begin{array}{l} X_1 = \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix} \\ X_2 = \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix} \\ X_3 = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \end{array}\right.$$

$\ker = \mathbb{Z}_2$

$\mathrm{im} = SO(3)$

$g \sim -g$ ↙ $SU(2)/\mathbb{Z}_2 \cong SO(3)$ $H = \mathbb{Z}_2$ $\mu_G \xrightarrow{?} \mu_{G/H}$

$\varphi \sim \varphi + 2\pi$

$\pi_H(f)(g) := \int_H f(gh)\,d\lambda(h)$ $h \in H$ a discrete normal subgroup

function on $G/H \cong SO(3)$

$$\int_{G/H} \pi_H(f)(g)\,d\nu_{G/H}(gH) = \int_G f(g)\,d\mu_G(g)$$

same form $\mu(g) = \cancel{C}\ \frac{1}{8\pi^2}\ \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\psi}_{2\pi \times 2 \times 2\pi = 8\pi^2}$

8: L & R Haar measure.

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R},\ x > 0 \right\}$$

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{y}{x} \\ 0 & 1 \end{pmatrix} \in G$$

$$\underbrace{\overset{g}{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}}}_{g_0} \underbrace{\overset{g_0}{\begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}}}_{g} = \begin{pmatrix} x\underline{u} & xv + y \\ 0 & 1 \end{pmatrix} \in G$$

① left: $g \mapsto g_0 g$

$du\,dv \mapsto x^2\,du\,dv$

Haar measure: $\underline{\int x^{-2} dx\,dy}$

② right: $g \mapsto g g_0$

$dx\,dy \mapsto u\,dx\,dy$

Haar measure: $\underline{\int x^{-1} dx\,dy}$

Proposition If (T, V) is rep of a compact group G, and V is an inner product space

$\Rightarrow$ (T, V) is unitarizable.

If T is not already unitary w.r.t in product $\langle \cdot \rangle_1$, then we can define a new inner product

$$\langle v, w \rangle_2 := \int_G \langle T(g) v, T(g) w \rangle_1 \, d\mu(g)$$

Then

$$\langle T(g) v, T(g) w \rangle_2 = \langle v, w \rangle_2$$

$$\langle T(h) v, T(h) w \rangle_2 = \int_G \langle T(hg) v, T(hg) w \rangle_1 \, d\mu(g)$$

$$\overset{\text{Haar}}{=} \int_G \langle T(g) v, T(g) w \rangle_1 \, d\mu(g)$$

$$= \langle v, w \rangle_2$$

Remarks:

1 det: $GL(n, K) \longrightarrow K^*$ non compact

$A \longmapsto \det A$

$$\langle \det A \, z_1, \det A \, z_2 \rangle = |\det A|^2 \, \bar{z}_1 z_2$$

skipped in class

2. Compact groups. $\exists A$ s.t. the matrix rep

$$u(g) = A\, M(g) A^{-1} \quad \forall g$$

where $u(g)$ unitary

Define usual inner product on $\mathbb{C}$

two set of basis $\{e_i^{1}\}$ $\{e_i^{2}\}$. $\{e^{2}\}$ is ON.

$$e_i^{(1)} = \sum_k A_{ki} e_k^{(2)}$$

$$\langle e_i^{(1)}, e_j^{(1)} \rangle = \sum_{kk'} \langle A_{ki} e_k^{(2)}, A_{k'j} e_{k'}^{(2)} \rangle$$

$$= \sum_{kk'} \bar{A}_{ki} A_{k'j} \delta_{kk'}$$

$$= \sum_k \bar{A}_{ki} A_{kj}$$

$$= (A^{+}A)_{ij}$$

if u is unitary w.r.t. $\{e_i^{(2)}\}$

then the unitary rep in $\{e_i^{(1)}\}$

is $\hat{u} = A^{-1} u A$

$$\hat{u}\, e_i^{(1)} = \sum_j \hat{u}_{ji} e_j^{(1)} = \sum_{jk} \hat{u}_{ji} A_{kj} e_k^{(2)}$$

$$= \sum_k (A\hat{u})_{ki} e_k^{(2)}$$

$$= \sum_k (uA)_{ki} e_k^{(2)}$$

$$\langle \hat{u} e_i^{(1)}, \hat{u} e_j^{(1)} \rangle = \sum_{kk'} \overline{(uA)_{ki}} (uA)_{k'j} \delta_{kk'}$$

$$= \sum_k [(uA)^{+}]_{ik} (uA)_{kj}$$

$$= (A^{+} u^{+} u A)_{ij}$$

$$= (A^{+}A)_{ij} = \langle e_i^{(1)}, e_j^{(1)} \rangle$$

8.6 The Regular representation.

Let G be a group. Then there is a left action of $G \times G$ on G:

$$(g_1, g_2) \longmapsto L(g_1) R(g_2^{-1}):$$

$$(g_1, g_2) \cdot g_0 = g_1 g_0 g_2^{-1}$$

and hence an induced action on $\mathrm{Map}(G, \mathbb{C})$

$$((g_1 \cdot g_2) \cdot f)(h) := f(g_1^{-1} h g_2)$$

which converts the vector space of functions $f : G \to \mathbb{C}$ into a represention space for $G \times G$.

Recall for induced g action:

$$\tilde{\phi}(g, F)(x) = F(\phi(g^{-1}, x))$$

$$\tilde{\phi}(g_1, \tilde{\phi}(g_2, F))(x) = \tilde{\phi}(g_2, F)(\phi(g_1^{-1}, x)) = F(\phi(g_2^{-1}, \phi(g_1^{-1}, x)))$$
$$= F(g_2^{-1} g_1^{-1} x)$$
$$= F(\phi((g_1 g_2)^{-1}, x))$$
$$= \tilde{\phi}(g_1 g_2, F)(x)$$

$$\{[(g_1, g_2)(g_3, g_4)] f\}(h) = [(g_1 g_3, g_2 g_4) f](h)$$
$$= f(g_3^{-1} g_1^{-1} h g_2 g_4)$$

$$\{(g_1, g_2) \cdot [(g_3, g_4) f]\}(h) = [(g_3, g_4) f](g_1^{-1} h g_2)$$
$$= f(g_3^{-1} g_1^{-1} h g_2 g_4)$$

This can be viewed as a group homomorphism

$$G \times G \to \mathrm{End}(\{\varphi\})$$

vector space $\varphi: G \to \mathbb{C}$ becomes a representation space for $G \times G$.

Now, equip G with a left and right-invariant Haar measure, and consider

$$L^2(G) = \{ f: G \to \mathbb{C} \mid \underbrace{\int_G |f(g)|^2 \, d\mu(g)}_{= \langle f, f \rangle} < \infty \}$$

i.e. the Hilbert space.

Then $G \times G$ action preserves the L^2-property because of the left & right Haar measure

<u>Definition</u> The representation $L^2(G)$ is known as the regular representation of G.

If we restrict $G \times G$ to subgroups $G \times \{1\}$ or $\{1\} \times G$. then $L^2(G)$ becomes a representation of G:

$$(L(h) \cdot f)(g) := f(h^{-1} g)$$

then it is the <u>left regular representation</u>

$$(R(h) \cdot f)(g) = f(gh)$$

defines the right regular representation

Note : $L(h)$, $R(h)$ acts on the function space on the left.

Example 1. $G = \mu_3 = \{1, \omega, \omega^2\}$ $\omega = e^{i\frac{2}{3}\pi}$

assign a basis in $L^2(G)$ $\delta_j(\omega^k) = \begin{cases} 1 & j = k \bmod 3 \\ 0 & \text{else} \end{cases}$

$$(L(\omega)\cdot\delta_0)(g) = \delta_0(\omega^{-1}g) = \delta_1(g)$$

$L(\omega)\delta_0 = \delta_1$

$L(\omega)\delta_1 = \delta_2$

$L(\omega)\delta_2 = \delta_0$

$$L(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$L(1)g = g$ $L(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{cases} L(\omega)1 = \omega \\ L(\omega)\omega = \omega^2 \\ L(\omega)\omega^2 = 1 \end{cases}$ $L(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

This is the "reg. rep" we talked before for finite groups.

Suppose (T, V) is a finite dim representation of G.

We can define $G \times G$ action on $\underline{\mathrm{End}(V) := \mathrm{Hom}(V, V)}$:

$\downarrow$

a representation space.

$\forall S \in \mathrm{End}(V)$:

$$(g_1 \cdot g_2) \cdot S := T(g_1) \cdot S \cdot T(g_2)^{-1}$$

How are the two representation space related?

For finite-dimensional V, we can define a map

$$\iota : \mathrm{End}(V) \longrightarrow L^2(G)$$

$$S \longmapsto f_S$$

$$f_S := \mathrm{Tr}_V (S T(g^{-1}))$$

V needs to be finite

which is $G \times G$ equivariant. (ι is an intertwiner)

$$\begin{tikzcd}
\mathrm{End}(V) \arrow[r, "\iota"] \arrow[d, "T_{\mathrm{End}(V)}"] & \mathrm{Map}(G, \mathbb{C}) \arrow[d, "T_{\mathrm{reg.\,rep}}"] \\
\mathrm{End}(V) \arrow[r, "\iota"] & \mathrm{Map}(G, \mathbb{C})
\end{tikzcd}$$

$$\begin{aligned}
\text{J}: (h_1, h_2) f_S(g) &= f_S(h_1^{-1} g h_2) \\
&= \mathrm{Tr}_V (S T(h_2^{-1} g^{-1} h_1)) \\
&= \mathrm{Tr}_V (S T(h_2)^{-1} T(g^{-1}) T(h_1)) \\
&= \mathrm{Tr}_V (T(h_1) S T(h_2)^{-1} T(g^{-1})) \\
&= \mathrm{Tr}_V ((h_1, h_2) S T(g^{-1})) \\
&= f_{(h_1, h_2) \cdot S} (g) = \text{L}
\end{aligned}$$

$$\begin{tikzcd}
& S \arrow[dl] \arrow[dr] & \\
(h_1, h_2) \cdot S \arrow[d] & & f_S \arrow[d] \\
f_{(h_1, h_2) \cdot S} & = & (h_1, h_2) f_S
\end{tikzcd}$$

Equip V with an ordered basis $\{v_i\}$

$$T(g)\cdot v_i = \sum_j M(g)_{ji} v_j$$

and take S to be the matrix unit e_{ij}

($[e_{ij}]_{ab} = \delta_{ia}\delta_{jb}$, a basis of $\mathrm{End}(V)$)

$$f_S = \mathrm{Tr}_V (S\, T(g^{-1}))$$

$$= \mathrm{Tr} (\sum_b \delta_{ia}\delta_{jb} M_{bc}(g^{-1}))$$

$$= \sum_{ac} [\delta_{ia} M_{jc}(g^{-1})]\, \delta_{ac}$$

$$= M_{ji}(g^{-1})$$

($f_S = M_{ij}(g)$ if replace V by its dual space V^*.

recall $M^*(g) = [M(g^{-1})]^{tr} = M(g)^{tr,-1}$)

$\Rightarrow$ f_S's are linear combinations of matrix elements of rep. of G.

(in other words. $M_{ij}(g) \in L^2(G)$ can be seen as basis)