8. Representation theory

8.1. Some motivation

1. In QM. symmetries are represented by unitary / linear, antiunitary / antilinear operators in Hilbert space $\mathcal{H}$.

(Wigner. 1931 ; Weinberg. QFT-I. 1995)

If the Hamiltonian H has certain symmetry. represented by U. $U^{+} H U = H$. / $[H. U] = 0$

They have the same eigen states.

$\Rightarrow$ simultaneous diagonalization.

$$H = t \sum_{\langle ij \rangle} C_i^{+} C_j + h.c.$$

$|i\rangle \langle i+1|$

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & t & & & & & t \\ t & 0 & t & & & & \\ & t & 0 & t & & 0 & \\ & & t & 0 & t & & \\ & & & t & 0 & t & \\ & 0 & & & t & 0 & t \\ t & & & & & t & 0 \end{pmatrix}$$

$HT = TH$

$$C_k^{+} = \sum_i e^{ikr_i} C_i^{+} \ / \ C_i^{+} = \sum_k e^{-ikr_i} C_k^{+}$$

$$\Rightarrow \tilde{H} = 2t \sum_{k_i} \cos k_i a \ C_k^{+} C_k$$

$$k_i = \frac{2\pi i}{aN} \qquad i = 0, \cdots N-1$$

$$\tilde{H} = \begin{pmatrix} 2t\cos k_1 & & \\ & 2t\cos k_2 & \\ & & \ddots \end{pmatrix}$$

eigen space labeled by $\underline{k_i}$

2 symmetry $\Leftrightarrow$ selection rules $[H, W] = 0$

$\Rightarrow \exists S$. s.t

block-diagonal

above, if $t_{12} = t' \neq t$

$\Rightarrow$ no longer block diagonal

$$S H S^{-1} = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix}$$

symmetry sectors labeled by (a set of) different quantum numbers

e.g. for Fermions . QN. = particle number
$\begin{cases} S \\ S_z \end{cases}$

$\begin{cases} [S_z, H] = 0 \\ [S^2, H] = 0 \end{cases}$

$|\uparrow\uparrow\rangle$ $|\uparrow\downarrow, -\rangle$ $|\downarrow, \uparrow\rangle$, $|\uparrow\downarrow\rangle$ $|-, \uparrow\downarrow\rangle$ $|\downarrow, \downarrow\rangle$

0

$$\begin{pmatrix} u & t & -t & 0 \\ t & 0 & 0 & t \\ -t & 0 & 0 & -t \\ 0 & t & -t & u \end{pmatrix}$$

0

$|S_z\rangle$

$\downarrow$

$|S, S_z\rangle$

$$\begin{pmatrix} u & -\sqrt{2}t & 0 \\ -\sqrt{2}t & 0 & -\sqrt{2}t \\ 0 & -\sqrt{2}t & u \end{pmatrix} \quad (0)$$

3. Conservation laws.

Noether's theorem:

continuous symmetry $\Leftrightarrow$ classically conserved current.

8.2 Review of basic definitions

① $G \to GL(V)$

V some vector space over field K

$GL(V)$ / $Aut(V)$: invertible linear transformations $V \to V$.

② rep. of G is a group homomorphism.

$$T : G \to GL(V)$$
$$g \mapsto T(g)$$

(T, V) denotes the representation. or T or V

$$T(g_1)T(g_2) = T(g_1 g_2) \quad \forall g_1, g_2 \in G.$$

$\dim V$: dim of rep.

V is called the carrier space / representation space.

mention red./irred here

> Given an ordered basis of finite dim. V.

$$\{\hat{e}_1, \cdots \hat{e}_n\} \Rightarrow GL(V) \cong GL(n, K)$$

$\begin{pmatrix} T_1(g) & R(g) \\ & T_2(g) \end{pmatrix}$ R arbitrary

$$\underline{T(g)\hat{e}_i = \sum_j M(g)_{ji}\hat{e}_j}$$

$$T(g_1)[T(g_2)\hat{e}_i] = T(g_1)\sum_j M(g_2)_{ji}\hat{e}_j$$
$$= \sum_j M(g_2)_{ji}(T(g_1)\hat{e}_j)$$
$$= \sum_j M(g_2)_{ji}\sum_k M(g_1)_{kj}\hat{e}_k$$
$$= \sum_k (M(g_1)M(g_2))_{ki}\hat{e}_k$$

$T(g_1)T(g_2) = T(g_1 g_2) \Leftrightarrow M(g_1)M(g_2) = M(g_1 g_2)$

In terms of group actions. rep. of G is a G-action on a vector space that respects linearity

$$g\cdot(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 g\cdot v_1 + \alpha_2 g\cdot v_2 \qquad \begin{matrix} v_i \in V \\ \alpha_i \in k \end{matrix}$$

Examples

1. rep. of degree / dim 1.

$$T: G \to \mathbb{C}^*$$

for element of order n. $g^n = 1_G$

$T(g)^n = 1$ $\quad$ $T(g)$ are roots of 1

$\mathbb{Z}_3 \cong \mu_3 \cong A_3 = \langle g \rangle$ $\qquad T(g) = \omega = e^{i\frac{2\pi}{3}} / e^{i\frac{4\pi}{3}}$

if take $T(g) = 1$ $\quad$ trivial representation (unit)

↑ trivial homo.

2. "regular representation" of a finite group.
(more to be discussed later)
正则表示

Let $\dim V = |G| = n$. with an ordered basis set $\{\hat{e}_g\}$ $(g \in G)$

$$T(g_1)\cdot \hat{e}_{g_2} = \hat{e}_{g_1 g_2}$$

V	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$e = (0, 0)$
$a = (1, 0)$
$b = (0, 1)$
$c = (1, 1)$

$$T_{reg} : \mathbb{Z}_2 \times \mathbb{Z}_2 \to GL(V) \quad (\dim V = 4)$$

$$V = \{\hat{e}_e, \hat{e}_a, \hat{e}_b, \hat{e}_c\}$$

$T(e)\hat{e}_g = \hat{e}_g$ $\qquad \underline{T(e) = \mathbb{1}_4}$

$$\begin{cases} \chi(T(e)) = \dim V = 4 \\ \chi(T(g \neq e)) = 0 \end{cases}$$

$$\begin{cases} T(a)\hat{e}_e = \hat{e}_a \\ T(a)\hat{e}_a = \hat{e}_e \\ T(a)\hat{e}_b = \hat{e}_c \\ T(a)\hat{e}_c = \hat{e}_b \end{cases}$$

$$T(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3. more generally. G acts on set X

$$x \mapsto gx$$

Let V be a vector space with basis $\{e_x\}$ $(x \in X)$

$$T(g)e_x = e_{gx}$$

permutation representation.

4. $G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ $\qquad T : G \to GL(\mathbb{C})$

$$n \mapsto a^n \quad (a \in \mathbb{C}^*)$$

$$n_1 + n_2 \to a^{n_1} \cdot a^{n_2} = a^{n_1 + n_2}$$

5. $G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$. $\quad T : G \longrightarrow GL(2, k)$

$$n \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

6. $G = GL(n, k)$ → one-dim. representation

$$T(g) := |\det g|^{\mu}$$

$$T(g_1 g_2) = |\det(g_1 g_2)|^{\mu} = |\det g_1|^{\mu} |\det g_2|^{\mu}$$
$$= T(g_1) T(g_2)$$

$\cancel{7}$. 1+1 dim Lorentz group

$$x^{0\prime} = \cosh\theta\, x^0 + \sinh\theta\, x^1$$

$$x^{1\prime} = \sinh\theta\, x^0 + \cosh\theta\, x^1$$

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = B(\theta) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

$\left(B(\theta) \in O(1, 1) = \{ A \mid A^T \eta A = \eta \},\ \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$$B(\theta_1) \cdot B(\theta_2) = B(\theta_1 + \theta_2)$$

Examples: Direct sum, tensor product, and dual representations

(T_1, V_1) and (T_2, V_2) are two reps of G

with $\dim V_1 = n$ and $\dim V_2 = m$, and basis

$\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_m\}$

① $V_1 \oplus V_2$. vector space of dim. $n+m$

with basis $\{(v_1, 0), (v_2, 0) \dots, (0, w_1), (0, w_2) \cdots\}$

rep on $V_1 \oplus V_2$: $\quad g \cdot (v, w) := (g \cdot v, g \cdot w)$ — G-action

$[(T_1 \oplus T_2)(g)](v \oplus w) := T_1(g) v \oplus T_2(g) w$. — rep.

mat. rep.

$$M_{T_1 \oplus T_2}(g) = \begin{array}{c} \begin{array}{cc} \quad n & \quad\quad m \end{array} \\ \left(\begin{array}{c|c} M_{T_1}(g) & 0 \\ \hline 0 & M_{T_2}(g) \end{array}\right) \end{array}$$

② $V_1 \otimes V_2$: vector space of dim $\underline{n \cdot m}$, basis

$\{v_i \otimes w_j : 1 \le i \le n,\ 1 \le j \le m\}$

$$\left(\sum_i a_i v_i\right) \otimes \left(\sum_j b_j w_j\right) = \sum_{ij} a_i b_j v_i \otimes w_j$$

rep on $V_1 \otimes V_2$:

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$$

$$[(T_1 \otimes T_2)(g)](v \otimes w) := T_1(g) v \otimes T_2(g) w$$

$$[(M_1 \otimes M_2)(g)]_{ia, jb} = (M_1(g))_{ij} (M_2(g))_{ab}$$

③ The dual vector space. $V^\vee$ (or V^*)

$\{$linear maps: $V \to k\} := \mathrm{Hom}(V, k)$

with $v_i^\vee$. $\quad v_i^\vee(v_j) = \delta_{ij}$

$\dim V^\vee = \dim V = n.$

rep on $V^\vee$: $\quad (g\cdot v_i^\vee)(v_j) = v_i^\vee(g^{-1}\cdot v_j)$ (induced action on function space)

natural pairing:

$$(g\cdot v_i^*)(g\cdot v_j) = v_i^*(g^{-1}\cdot g\cdot v_j)$$
$$= v_i^*(v_j) = \delta_{ij}$$

$$T(g): V \to V, \quad v \mapsto T(g)v$$
$$T^\vee(g): V^\vee \to V^\vee, \quad v^\vee \mapsto T^\vee(g)v^\vee$$

$$\underline{v_j = \sum_i M_{ij} v_i}$$

$$\underline{v_i^\vee(v_j)} = \sum_k M^\vee(g)_{ki} v_k^\vee \left(\sum_l M(g)_{lj} v_l\right)$$
$$= \sum_{kl} M^\vee(g)_{ki} M(g)_{lj} \underbrace{v_k^\vee(v_l)}_{\delta_{kl}}$$
$$= \underline{\sum_l M^\vee(g)_{li} M(g)_{lj} = \delta_{ij}}$$

$$\Leftrightarrow \quad M^\vee(g) = [M(g^{-1})]^{tr} = M(g)^{tr.-1}$$

8.3 Equivalent reps and characters

<u>Definition</u>. Let (T_1, V_1) and (T_2, V_2) be two reps. of a group G. An intertwiner (intertwining map 交结映射) between these two reps is a <u>linear</u> transformation

$A: V_1 \to V_2$

s.t. $\forall g \in G$. the following diagram commutes.

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_2 \\ T_1(g)\downarrow & & \downarrow T_2(g) \\ V_1 & \xrightarrow{A} & V_2 \end{array}$$

i.e. $T_2(g)A = A \cdot T_1(g)$

A is an <u>equivariant</u> linear map of G spaces $V_1 \to V_2$

$\alpha A_1 + \beta A_2 \in \mathrm{Hom}_G$ $\mathrm{Hom}_G(V_1, V_2)$: vector space of all intertwiners.

<u>Definition</u>. Two reps $(T_1 . V_1)$ and $(T_2 . V_2)$ are equivalent $(T_1 . V_1) \cong (T_2 . V_2)$ if there is an intertwiner $A: V_1 \to V_2$ which is an <u>isomorphism</u>, that is

$$T_2(g) = A\, T_1(g) A^{-1} \quad (\forall g \in G)$$

For any finite-dimensional representation

$$T : G \to \mathrm{Aut}(V)$$

of any group G. we can define the character of the representation χ_T

$$\chi_T : G \to k$$

$$\chi_T(g) := \mathrm{Tr}_V(T(g))$$

1. equivalent $\Leftrightarrow$ same character function

$\chi_T(h^{-1}gh) = \chi_T(g)$ "class function"

2. independent of basis choices

3. For above representations.

a. $M_{T_1 \oplus T_2}(g) = \begin{pmatrix} M_{T_1}(g) & 0 \\ 0 & M_{T_2}(g) \end{pmatrix}$

$$\chi_{T_1 \oplus T_2} = \chi_{T_1} + \chi_{T_2}$$

b. $(M_1 \otimes M_2)(g)_{ia,jb} = (M_1(g))_{ij}\,(M_2(g))_{ab}$

8.4 Unitary representations

Let V be a complex vector space over $\mathbb{C}$.

Define the <u>inner product</u> on V as a sesquilinear

map $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{C}$. obeying

(1) $\langle v , * \rangle$ is linear for all fixed v.

(2) $\langle w , v \rangle = \overline{\langle v , w \rangle}$

(3) $\langle v , v \rangle \geq 0$. $= 0$ iff $v = 0$

sesquilinear:

$$\left(\begin{aligned} &\langle v , \alpha_1 w + \alpha_2 w_2 \rangle = \alpha_1 \langle v_1 , w_1 \rangle + \alpha_2 \langle v_1 , w_2 \rangle \\ &\langle \alpha_1 v_1 + \alpha_2 v_2 , w \rangle = \overline{\alpha_1} \langle v_1 , w \rangle + \overline{\alpha_2} \langle v_2 , w \rangle \end{aligned} \right)$$

<u>Definition</u>. Let V be an inner product space

A unitary rep is a rep (V, U)

s.t. $\forall g \in G$ $U(g)$ is a unitary

operator on V. i.e.

$\langle U(g) v , U(g) w \rangle = \langle v , w \rangle$ $\forall v, w \in V$

$\forall g \in G$.

<u>Definition</u>. If a rep (V, T) is equivalent to

a unitary rep. then it is said

to be <u>unitarizable</u>.

Consider a finite group. Let $T(g)$ be a (non-unitary) rep. To unitarize $T(g)$, define

(Dresselhaus 陶瑞宝)

$$H = \sum_{g \in G} T^{+}(g)\, T(g)$$

H is Hermitian and positive definite.

$$T^{+}(h)\, H\, T(h) = \sum_{g} T^{+}(h) T^{+}(g) T(g) T(h) = \sum_{g} T^{+}(gh) T(gh) = H$$

$\exists\, U$ s.t. $U^{+} H U = \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ $\quad (\forall \lambda_i > 0)$

Define $\tilde{T}(g) = \Lambda^{\frac{1}{2}} U^{+} T(g) U \Lambda^{-\frac{1}{2}}$

$$\tilde{T}^{+}(g)\, \tilde{T}(g) = (\Lambda^{-\frac{1}{2}} U^{+} \underbrace{T^{+}(g) \underbrace{U \Lambda^{\frac{1}{2}})(\Lambda^{\frac{1}{2}} U^{+}}_{H} T(g)}_{H} U \Lambda^{-\frac{1}{2}})$$

$$= \Lambda^{-\frac{1}{2}} \underbrace{U^{+} H U}_{\Lambda} \Lambda^{-\frac{1}{2}} = \mathbb{1}$$

$\Rightarrow \tilde{T}(g) = A^{-1} T(g) A \qquad A = U \Lambda^{-\frac{1}{2}} \qquad (\forall g)$

$\Rightarrow$ Representations of finite groups are equivalent to unitary representations (uniterizable)

$\Rightarrow$ What about continuous / infinite groups ?

Some ideas : $\sum_{g \in G} \to \int_G dg$?

$\hookrightarrow$ Haar measure
(later)

8.5 Haar measure (aka invariant integration)

Consider a function $f : G \to \mathbb{C}$. $f \in Map(G, \mathbb{C})$

$$\langle f \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \Longrightarrow \text{"} \underline{\int_G} dg\, f(g) \text{"}$$

$$\int_G dg \in (Map(G, \mathbb{C}))^\vee = Hom(Map(G, \mathbb{C}), \mathbb{C})$$

$$\int_G dg : f \mapsto \langle f \rangle$$

For finite group. $\underline{\frac{1}{|G|} \sum_{g \in G} f(hg) = \frac{1}{|G|} \sum_{g \in G} f(g)}$

invariant under left translation $L_h : g \mapsto hg$

We require similarly for $\int_G dg$:

$$\underline{\int_G f(hg) dg = \int_G f(g) dg} \quad (\forall h \in G)$$

left invariance condition.

Left Haar measure.

(right Haar measure : $\int_G f(gh) dg = \int_G f(g) dg$)

1. For a finite group left and right invariant measures are unique up to an overall scale.

holds also for compact Lie groups.

in general physics context: subset of $\mathbb{C}^m$.

compact $\Leftrightarrow$ closed & bounded

$U(n) = \{ A \in GL(n, \mathbb{C}) \mid A^+ A = \mathbb{1} \} \subset \mathbb{C}^{n^2}$

$\sum_j (A^+)_{ij} A_{ji} = 1$

$\Rightarrow \sum_j |A_{ji}|^2 = 1 \Rightarrow |A_{ji}| \leq 1 \quad \forall i,j$

other examples: $Sp(n) \cong U(2n) \cap Sp(2n, \mathbb{C})$

$Sp(1) \cong SU(2)$

non-compact: $O(1, d)$

$Sp(2n, \mathbb{K}) \to \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \quad B^T = B$

$GL(n, \mathbb{K})$

Examples

1. $G = \mathbb{R}$

$$\int_G dg\, f(g) = \int_G dg\, f(g+a) \qquad (a \in \mathbb{R})$$

$$\Rightarrow c\int_{-\infty}^{+\infty} dx\, f(x)$$

$$c\int_{-\infty}^{\infty} dx\, f(x+a) = c\int_{-\infty}^{\infty} d(x+a)\, f(x+a) = c\int_{-\infty}^{\infty} dx\, f(x)$$

2. $G = \mathbb{Z}$

$$\int_G dg\, f(g) = c\sum_{n\in\mathbb{Z}} f(n)$$

3. $G = \mathbb{R}^*_{>0}$

$$\int_G f(g)\, dg = c\int_0^{\infty} f(x)\,\frac{dx}{x}$$

$\forall a \in \mathbb{R}^*_{>0}$: $\underline{\int_0^{\infty} f(ax)\,\frac{dx}{x}} = \int_0^{\infty} f(x)\,\frac{d(x/a)}{x/a} = \underline{\int_0^{\infty} f(x)\,\frac{dx}{x}}$