Review of Group part

1. Definition of groups $(G, e, \underline{m}, \underline{I})$

① set G

② $e \in G$ $\quad e g = g \cdot e = g$

③ $m: G \times G \to G$

④ $I: G \to G$

$G = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ $\quad$ groups if $m =$ "+"

not if $m =$ "×"

$\mathbb{R}^* = \mathbb{R} - \{0\}$, $\quad \mathbb{C}^* = \mathbb{C} - \{0\}$

↪ $|G|$ order $\begin{cases} \text{finite} \\ \text{infinite} \end{cases}$ group

↪ $g_1 g_2 = g_2 \cdot g_1 \quad \forall g_1, g_2 \to$ abelian

↛ noabelian.

2. Direct product $H \times G$.

$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot h_2, g_1 \cdot g_2)$

↪ semidirect product $H \rtimes_\alpha G$. $h \in H$. $g \in G$.

$$(\underline{h_1}, \underline{g_1}) \cdot_\alpha (\underline{h_2}, \underline{g_2}) = (\underline{h_1 \alpha_{g_1}(h_2)}, \underline{g_1 g_2})$$

$$\{R_1 | \tau_1\}\{R_2 | \tau_2\} \vec{r} = \{R_1 | \tau_1\} (R_2 \vec{r} + \vec{\tau}_2)$$

$$= R_1 R_2 \vec{r} + R_1 \vec{\tau}_2 + \vec{\tau}_1$$

$$(\tau_1, R_1)(\tau_2, R_2) = (R_1 \vec{\tau}_2 + \vec{\tau}_1, R_1 R_2)$$

↪ symmorphic space groups

3. subgroups $H \subset G$.

$\underline{m}$: $H \times H \to H$

$\underline{I}$: $H \to H$

G has trivial subgroups $\{e\}$ and G.

proper subgroup $H \neq G$

$\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ "+"

↪ $H \triangleleft G$: $gHg^{-1} = H$ $(\forall g \in G)$

↪ simple group, no nontrivial normal subgroup

↪ centralizer $C_G(h) = \{g \in G. \ gh = hg\} \subset G$

$C_G(H) = \{g \in G. \ gh = hg, \forall h \in H\} \subset G$

normalizer $N_G(H) = \{g \in G. \ gHg^{-1} = H\}$

$$C_G(H) \subset N_G(H)$$

4. $GL(n, k)$

$\hookrightarrow SL(n, k)$

$O(n, k)$, $SO(n, k)$ } $\underline{\det}$

$U(n, k)$, $SU(n, k)$

$A^T \underline{J} A = \underline{J}$ { $J_{p,q} = \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix}$

symplectic $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Homomorphism / isomorphism.

5. homomorphism. $\varphi : G \longrightarrow G'$

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \varphi \times \varphi \downarrow & & \downarrow \varphi \\ G' \times G' & \longrightarrow & G' \end{array}$$

$$\varphi(g_1) \cdot_{G'} \varphi(g_2) = \varphi(g_1 \cdot_G g_2)$$

$$\begin{cases} \varphi(e) = e' \\ \varphi(g^{-1}) = \varphi(g)^{-1} \end{cases}$$

ker / im : $\ker \varphi = \{ g \in G : \varphi(g) = 1_{G'} \}$

$\operatorname{im} \varphi = \varphi(G)$

① $\pi: SU(2) \longrightarrow SO(3)$

$$u\, \vec{x}\cdot\vec{\sigma}\cdot u^{+} := (\pi(u)\cdot\vec{x})\cdot\vec{\sigma}$$

$$\ker \pi = \{\pm 1\} \cong \mathbb{Z}_2$$

② $\Gamma: G \to \underline{GL(V)}$ V some $\overset{n\text{-dim}}{\checkmark}$ vector space over k

given basis

$$GL(V) \cong \underline{GL(n.k)}$$

<u>isomorphism</u>: homo. + (1-1 & onto)

1-1: $\ker\varphi = \{e\}$

onto: $\varphi(G) = G'$

$\varphi: G \to G$: $\mathrm{Aut}(G)$

isomorphism defines an equivalence relation

$$\mu_N \cong \mathbb{Z}_N$$

<u>matrix-rep.</u> $T: G \to GL(n.k)$

$$T(g)\hat{e}_i = T(g)_{ij}\,\hat{e}_j$$

↳ equivalent rep $T \cong T'$, $\exists S$ s.t.

$$T'(g) = S\,T(g)\,S^{-1} \quad \forall g \in G.$$

more generally <u>conj. rep</u> $\varphi_{1,2}: G \to G'$

$$\varphi_2(g_1) = g_2\,\varphi_1(g_1)\,g_2^{-1}$$

6. define group action by homomorphism.

$$\Phi : G \longrightarrow S_X := \{ X \xrightarrow{f} X \}$$

set of permutations

$$g \longmapsto \phi(g, \cdot)$$

$$\Phi_g(x) = \phi(g, x) = g \cdot x$$

$$g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) x$$

↳ orbits. $Orb_G(x) = \{ g \cdot x . g \in G \}$

① defines equivalence relation

$x \sim y . \quad y = g \cdot x$

② orbits partition G.

$$O_G(x) = O_G(x') \text{ or}$$

$$O_G(x) \cap O_G(x') = \phi.$$

X/G set of orbits

↳ fixed points

$$Fix_X(g) = \{ x \in X : gx = x \} \subset X$$

→ stabilizer.

$$Stab_G(x) := \{ g \in G . g \cdot x = x \} \subset G.$$

(G^x)

6. group action.

orbits, fixed points, stabilizer

Theorem. (Stab - orbit)

$$O_G(x) \xrightarrow{\cong} G/G^x$$

$$|O_G(x)| = [G : G^x]$$

$SO(3)$ acts on S^2. $\quad Orb_{SO(3)} = S^2$

$Stab_{SO(3)}(\hat{z}) \cong SO(2)$

$$S^2 \cong SO(3)/SO(2)$$

$SU(2)$ on $\mathbb{C}^2$. $\quad S^3 \cong SU(2)$

7. G - action on itself.

① H a subgroup, right action on G.

$gH = \{gh.\ h \in H\}$ left - cosets

$|gH| = |H|$

+ Lagrange. Finite G.

$$|G|/|H| = [G : H]$$

② action by conjugation.

orbits / conjugacy class

$C(h) = \{ \underline{g h g^{-1}} \ g \in G \}$

Stab-orb. $|C(g)| = [G : C_G(g)]$

$\hookrightarrow \mathrm{Stab}_G(g)$

<u>centralizer</u>

Finite G. $\left.\begin{aligned} |C(g)| &= \frac{|G|}{|C_G(g)|} \\ + \ |G| &= \Sigma |C(g)| \end{aligned}\right\}$

$\Rightarrow \ |G| = \sum_{\{(g)\}} \frac{|G|}{|C_G(g)|}$ "class equation"

$\hookrightarrow$ ① $|G| = p^n \Rightarrow Z(G) \neq \{e\}$

② (Cauchy) $p \mid |G|$

$\Rightarrow \exists g \in G . \ g^p = 1$

$\hookrightarrow$ <u>class function</u>.

function f on G.

$f(g g_0 g^{-1}) = f(g_0) \quad \forall g_0 . g \in G$.

$\hookrightarrow$ mat rep.

$\chi_T(g) = \mathrm{Tr}\, T(g)$ character

$\hookrightarrow$ equivalent rep. $\varphi_1 . \varphi_2$ $\exists g_2$. s.t.

$\varphi_2(g) = g_2 \varphi_1(g) g_2^{-1} \quad \forall g_1 \in G$.

mat. rep. $T_1 : G \to GL(n, k)$

$T_2 : G \to GL(n, k)$

$\exists S \in GL(n, k)$ s.t.

$T_2(g) = S T_1(g) S^{-1} \quad \forall g \in G$.

8. Morphisms of G spaces / equivariant map

$f : X \to X'$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi(g)"'] & X' \arrow[d, "\Phi'(g)"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

$f(\Phi(g, x)) = \phi'(g, f(x))$

$f(gx) = g \cdot f(x)$

9. The symmetric group S_n.

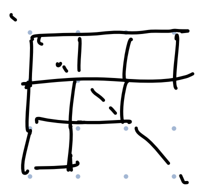
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = (1243)$

① unique cycle decomposition of $\phi \in S_n$

② r-cycles are conjugate

↳ conjugacy classes labeled by partitions of n.

S_6. $\vec{\lambda} = \{3, 2, 1\}$



Young diagram.

$\mathrm{sgn}: S_n \longrightarrow \mathbb{Z}_2$

$\phi \longmapsto \mathrm{sgn}(\phi) = (-1)^{n-t}$ → len. of cycle decomposition.

$A_n \triangleleft S_n \qquad \mathrm{sgn}(\phi \in A_n) = 1$

Why S_n?

finite G of order n.

$\xrightarrow{\text{embed}} S_n$

$\cong$ some subgroup of S_n

$D_4 \cong S_4 \subset S_8 \quad (|D_4| = 8)$

10. quotient groups

$N \triangleleft G$. then G/N has a natural group structure

$(g_1 N)\cdot(g_2 N) := (g_1 g_2)\cdot N.$

$\mu: G \longrightarrow G/N.$

$g \longmapsto gN$

$\ker\mu = N.$

1st. isomorphism theorem $\quad \mu: G \to G'$

$G/\ker\mu \cong \mathrm{im}\,\mu.$

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11. exact sequence.

$$\to G_{i-1} \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1}$$

$$\operatorname{im} f_{i-1} = \ker f_i$$

SES. $1 \xrightarrow[f_0]{} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow[f_3]{} 1$

① $\ker f_1 = \operatorname{im} f_0 = \{1\}$ f_1 injective

② $\operatorname{im} f_2 = \ker f_3 = G_3$ f_2 surjective.

$$1 \to N \to G \to Q \to 1$$

G is an extension of Q by N.

$$\begin{cases} N \cong H \triangleleft G \\ Q \cong G/N \end{cases}$$

↳ Central extension. $N \subset Z(G)$

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1$$