3. $\det : \quad O(n) \longrightarrow \mathbb{Z}_2 \qquad AA^T = \mathbb{1} \Rightarrow \det = \pm 1$

$\qquad M \longmapsto \det(M)$

$\ker(\det) = SO(n)$

$$1 \rightarrow SO(n) \longrightarrow O(n) \longrightarrow \mathbb{Z}_2 \rightarrow 1$$

4. $\det \quad U(n) \longrightarrow U(1)$

$\qquad u \longmapsto \det u = \lambda \qquad \ker ? \quad SU(n)$

$$1 \rightarrow SU(n) \rightarrow U(n) \longrightarrow U(1) \rightarrow 1$$

5. $\pi : SU(2) \longrightarrow SO(3)$

$\qquad u\,\vec{x}\cdot\vec{\sigma}\,u^{+} := (\pi(u)\vec{x})\cdot\vec{\sigma}$

$u \in \ker\pi . \quad u\,\vec{x}\cdot\vec{\sigma}\,u^{+} = \vec{x}\cdot\vec{\sigma} \qquad u = \lambda\mathbb{1}$

$\lambda = \pm 1$

$\pi(u) = \pi(-u)$

$\Rightarrow \ker\pi \cong \mathbb{Z}_2$

$SU(2)/\mathbb{Z}_2 \cong SO(3)$

$$1 \rightarrow \mathbb{Z}_2 \longrightarrow SU(2) \longrightarrow SO(3) \longrightarrow 1$$

$\uparrow$

abelian. and $\mathbb{Z}_2 \subset Z(SU(2))$

Definition (central extension)

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

1. A is abelian.
2. $A \subset Z(E)$ $\quad i(a)\, b = b\, i(a)$ $(\forall a \in A, \forall b \in E)$

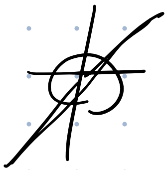
$$1 \to N \to G^{\text{Quantum}} \to G^{\text{classical}} \to 1$$

Motivation for such extensions in QM §11.1 §15.1 & 2

In QM we talk about $\mathcal{H}\setminus\{0\}$ but distinct states are rep.ed by a set of vectors:

$$|\psi_1\rangle \sim |\psi_2\rangle \quad \text{iff} \quad |\psi_1\rangle = \lambda |\psi_2\rangle . \ \lambda \in \mathbb{C}^*$$

"rays" $\underline{\psi} = \{e^{i\alpha}\psi : \alpha \in \mathbb{R}\}$



This is actually the projective complex plane. $(z_1, z_2, z_3) \sim \lambda(z_1, z_2, z_3)$, $(\lambda \in \mathbb{C}^*)$ $\in \mathbb{CP}^3$

represented by $[z_1 : z_2 : z_3]$

This is not a linear space:

$[1:0:0] \overset{?}{+} [0:1:0]$ cannot be defined.

no zero vector $[0:0:0]$

uniquely defined by the density matrix

$$\rho_\varphi = \frac{|\varphi\rangle\langle\varphi|}{\langle\varphi|\varphi\rangle} \qquad (\rho^2=\rho)$$

projective Hilbert space $P\mathcal{H} := (\mathcal{H}\setminus\{0\})/\sim$

Consider symmetry operations on $P\mathcal{H}$.

The overlap. $o(\rho_1,\rho_2) = \mathrm{Tr}(\rho_1\rho_2) = \dfrac{|\langle\varphi_1|\varphi_2\rangle|^2}{\|\varphi_1\|^2\|\varphi_2\|^2}$

should be conserved.

But for all kinds of reasons we want to work on liner spaces $\mathcal{H}$. Aut($\mathcal{H}$)

[Wigner's theorem. sym. operations are unitary or antiunitary

$$\begin{cases} \langle U\varphi, U\phi\rangle = \langle\varphi, \phi\rangle \\ \langle A\varphi, A\phi\rangle = \overline{\langle\varphi,\phi\rangle} = \langle\phi, \varphi\rangle \end{cases}$$

$(\ Ai = -iA\ ,\ A = UK)$

↪ cplx. conj]

Now consider unitary symmetry operations on states.

one only needs

$$U(g_1)U(g_2)\varphi = \underline{\alpha(g_1,g_2)}\, U(g_1g_2)\varphi$$

$\alpha : G \times G \to U(1)$

Not quite a group representation. projective. rep.

$$\left(\begin{aligned} &\text{if } \tilde{u}(g) = b(g)\, u(g) \\ &\quad \Rightarrow b(g_1)\, b(g_2) = f(g_1, g_2)\, b(g_1 . g_2) \quad \forall g_1 . g_2 . \end{aligned} \right)$$

reduce back to rep.

rotation of spins:

SU(2) is a spin projective rep of SO(3)

classical $\qquad$ quantum

$SO(3) \longrightarrow SU(2)$

Euler angles (zxz)

$$R(\phi . \theta . \psi) \to e^{i \frac{\phi}{2} \sigma^3} e^{i \frac{\theta}{2} \sigma^1} e^{i \frac{\psi}{2} \sigma^3}$$

$R(2\pi . 0 , 0) = 1 .$ -> ± 1 for fermion / boson

a $\mathbb{Z}_2$ phase

The central extension of G by A

$$1 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$$

is classified by the 2-cohomology group $H^2(G . A)$

6. finite Heisenberg group.

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} \omega & & & 0 \\ & \omega^2 & & \\ & & \omega^2 & \\ 0 & & & 1 \end{pmatrix}$$

$$\omega = e^{i \frac{2\pi}{4}}$$

$$\underline{\underline{QP = \omega PQ}} \quad \leftarrow$$ Weyl relation of canonical commutation relation.

Some background.

$[q, p] = i\hbar \quad (\hbar = 1)$

$qp - pq = i \qquad p, q$. acts on $f(\mathbb{R})$

$\Rightarrow \quad A = e^{i\xi p} \quad B = e^{i\eta q} \qquad$ (Weyl.)

$AB = e^{i\xi p} \cdot e^{i\eta q} \qquad e^X e^Y = e^Z$

$Z = X + Y + \frac{1}{2}[X, Y]$

$$\left\{ \begin{aligned} &= e^{i(\xi p + \eta q) + \frac{1}{2}[i\xi p, i\eta q]} \\ BA &= e^{i(\xi p + \eta q) + \frac{1}{2}[i\eta q, i\xi p]} \end{aligned} \right.$$

$+ \frac{1}{12}\left([X, \underline{[X, Y]}] - [Y, [X, Y]]\right)$

$+ \cdots$

$\Rightarrow \quad AB = e^{i\xi\eta} BA \equiv \omega BA \quad (A, B : n \times n$ mats.)

$\det(AB) = \omega^n \det(BA) \Rightarrow \underline{\omega^n = 1}$

$$\left\{ \begin{aligned} A^k B &= \omega^k B A^k \\ A B^l &= \omega^l B^l A \end{aligned} \right. \rightarrow A^k B^l = \omega^{kl} B^l A^k$$

$k=n \cdot l=1 \Rightarrow \underline{A}^n B = \overset{1}{\cancel{\omega^n}} B A^n \overset{?}{\Longrightarrow} A^n = \mathbb{1}$

similary $B^n = \mathbb{1}$.

A general element in $\underline{\text{Heis}_N}$ has the form

$$\omega^a P^b Q^c$$

$$(\omega^{a_1} P^{b_1} Q^{c_1}) \cdot (\omega^{a_2} P^{b_2} Q^{c_2}) = \omega^{a_3} P^{b_3} Q^{c_3}$$

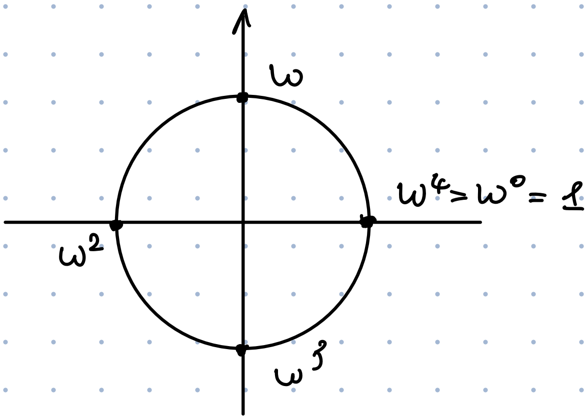
$$\begin{cases} a_3 = a_1 + a_2 + c_1 b_2 \\ b_3 = b_1 + b_2 \\ c_3 = c_1 + c_2 \end{cases}$$

$$\pi : \text{Heis}_N \longrightarrow \mathbb{Z}_N \times \mathbb{Z}_N$$

$$\omega^a P^b Q^c \longmapsto (b \bmod N,\ c \bmod N)$$

$$\ker(\pi) = \{ \omega^a \overset{\mathbb{1}}{\cancel{P^{rN}}} \overset{\mathbb{1}}{\cancel{Q^{sN}}} \} \cong \mathbb{Z}$$

$$1 \longrightarrow \mathbb{Z}_N \longrightarrow \text{Heis}_N \longrightarrow \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \underline{1}$$



$(P \cdot \Psi)(\omega^k) := \Psi(\omega^{k-1})$ translation

$(Q\Psi)(\omega^k) := \omega^k \Psi(\omega^k)$ position operator

$$(QP)\Psi(\omega^k) = \omega^k P\Psi(\omega^k) = \omega^k \Psi(\omega^{k-1})$$

$$\{ (PQ)\Psi(\omega^k) = Q\Psi(\omega^{k-1}) = \omega^{k-1}\Psi(\omega^{k-1})$$

$\Rightarrow QP = \omega PQ$

$N \to \infty$: $\mathbb{Z}_N \to U(1)$

$\mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{R} \times \mathbb{R}$

$$1 \to U(1) \to \mathrm{Heis}(\mathbb{R} \times \mathbb{R}) \to \mathbb{R} \times \mathbb{R} \to 1$$

7. More on group actions

7.1. Some defs and S-O theorem

Recall that the group action of G on a set X:

$$\phi : G \times X \to X$$

① left action. $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$$

(right action $(x \cdot g_2) \cdot g_1 = x \cdot (g_2 g_1)$)

② $\phi(1_G, x) = x$

mention different forms of L & R actions and induced actions on $\mathcal{F}[X \to Y]$

A G-action is: see Moore's note

① effective : $\forall g \neq 1, \exists x . \text{ s.t. } gx \neq x$

(ineffective $\exists g \neq 1, \forall x . \text{ s.t. } gx = x$)

② transitive : $\forall x, y \in X . \ \exists g . \text{ s.t. } \underline{y = g \cdot x}$

there is only one orbit

③ free : $\underline{\forall g \neq 1} . \ \underline{\forall x} . \ \underline{g \cdot x} \neq x$

Definitions.

1. isotropy group (stabilizer group

$$\mathrm{Stab}_G(x) := \{ g \in G . \ g \cdot x = x \} \subset G$$

$(\equiv G^x)$

$$\left(\begin{array}{c} g_1 . \ g_2 \in G^x \\ g_1 \cdot x = x \quad g_2 (g \cdot x) = g_2 x = x \\ g_2 g_1 \in G^x \end{array} \right)$$

If the group action of G is free

$\Leftrightarrow \ G^x = \{ 1 \} \quad \forall x \in X.$

2. If $\exists g (\in G^x) \neq 1 \quad g \cdot x = x$. x is called a fixed point.

$$(X^g \equiv) \, \mathrm{Fix}_X(g) = \{ x \in X : g \cdot x = x \} \subset X$$

is the fixed point set of g.

free $\Leftrightarrow X^g = \phi \quad (g \neq 1)$

3. $O_G(x) = \{ gx . \ \forall g \in G \}$

Theorem (Stabilizer - orbit)

Let X be a G-set. Each left-coset of G^x $(\equiv \mathrm{Stab}_G(x))$ $(x \in X)$ is in a natural

1 - 1 correspondence with points in $O_G(x)$.

There exists a natural isomorphism

$$\varphi : O_G(x) \longrightarrow G/G^x$$
$$g\cdot x \longmapsto g\cdot G^x$$

① Well defined.

$$gx = g'x$$
$$\Leftrightarrow (g'^{-1}g)x = x \Leftrightarrow g'^{-1}g \in G^x \Leftrightarrow gG^x = g'G^x$$

② surjective ✓

injective : $g\cdot G_x = g'\cdot G_x \Rightarrow gx = g'x$

For a finite group : $|O_G(x)| = [G : G^x] = |G|/|G^x|$

Example .

1. G acts on G by conjugation $h \in G$.

$$O_G(h) = \{ghg^{-1}, \forall g \in G\} = C(h)$$

$$\mathrm{Stab}_G(h) \equiv G = \{g \in G.\ \underline{ghg^{-1} = h}\} =: C_G(h)$$

Definition . The centralizer of h in G

$$C_G(h) := \{g \in G : gh = hg\}$$

(1) $C_G(h)$ is a subgroup

① $e \in C_G(h)$: $eh = he$

② $\forall g_1, g_2 \in C_G(h)$ $(g_1 g_2^{-1})h = g_1 h g_2^{-1} = h g_1 g_2^{-1}$

$\Rightarrow (g_1 g_2^{-1}) \in C_G(h)$

$|C(h)| = [G : C_G(h)]$

↑

number of conjugates of h

extend to subsets.

$C_G(H) = \{ g \in G \cdot gh = hg \ \forall h \in H \}$

$C_G(G) = Z(G)$

7.2 Practice with terminology of group actions.

1. $X = \{1, \dots, n\}$. $G = S_n$

① effective. ✓ ($\forall \phi \neq 1. \underline{\underline{\exists}} x. \phi \cdot x \neq x$)

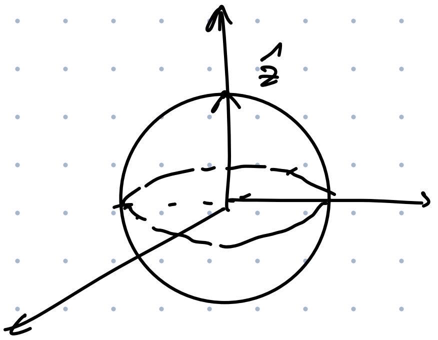
② transitive ✓

③ free × ($\forall \phi \neq 1. \underline{\underline{\forall}} x. \phi \cdot x \neq x$)

keep j fixed. $\veebar S_{n-1}$

$\{ S_{n-i} \}$

2. $SO(3)$ acts on S^2



① effective. ✓

② transitive ✓

③ free ? ✗

$$\mathrm{Stab}_{SO(3)}(\hat{z}) = \left\{ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \phi \in [0, 2\pi) \right\}$$

$$\cong SO(2)$$

$$\underbrace{\mathrm{Orb}_{SO(3)}(\hat{n})}_{\overset{\shortparallel}{S^2}} \cong SO(3)/SO(2)_{\hat{n}}$$