6.2 Conjugacy

Definition (a) a group element h is conjugate to h'

$\exists g \in G$. s.t. $h' = g h g^{-1}$

(b) conjugacy defines an equivalence relation.

The equivalence class is called the

conjugacy class (of h)

$C(h) := \{ g h g^{-1} : \forall g \in G \} \left(=: h^G \right)$

(c) $H \subset G$ is a subgroup. its conjugate

$H^g := g H g^{-1} = \{ g h g^{-1} : h \in H \}$ is also a subgroup

① $e \in H^g$ $\quad g e g^{-1} = e$

② $(g h_1 g^{-1})(g h_2 g^{-1}) = g(h_1 h_2) g^{-1} \in H^g$

③ $I(g h_1 g^{-1}) = g h_1^{-1} g^{-1} \in H^g$

Example

1. Permutations ϕ_1, ϕ_2 are conjugate if they have the same cycle decomposition structure.

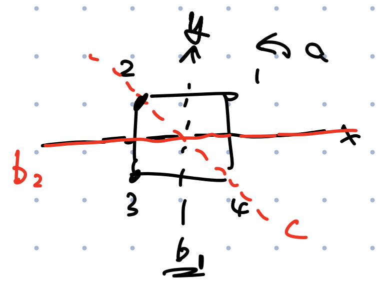
$$(a_1 a_2)(a_3 a_4 a_5) \sim (b_1 b_2)(b_3 b_4 b_5)$$

$$\Rightarrow \tau(a_1 a_2 \cdots a_n)\tau^{-1} = (\tau(a_1), \tau(a_2), \cdots \tau(a_n))$$

$$\tau(a_1 a_2)(a_3 a_4 a_5)\tau^{-1} = (b_1 b_2)(b_3 b_4 b_5)$$

$$\Leftrightarrow \boxed{\tau(a_i) = b_i}$$

2. $D_4 := \langle a, b : a^4 = b^2 = 1, (ab)^2 = 1 \rangle$



$a = (1234)$

$b_1 = (12)(34)$

$c = ab = (1234)(12)(34) = (13)(2)(4) = (13)$

$$cb_1c^{-1} = (13)((12)(34))(13) = (14)(23) = b_2 \qquad b_1 \sim b_2$$

$$D_4 = \{1\} \cup \{a^2\} \cup \{a, a^3\} \cup \{b, a^2b\} \cup \{ab, a^3b\}$$

$$= \{()\} \cup \{(13)(24)\} \cup \{(1234), (1432)\}$$

$$\cup \{(12)(34), (14)(23)\} \cup \{(13), (24)\}$$

$$\left(\begin{array}{l} \tau((13)(24))\tau^{-1} = ((12)(34)) \\ \tau(3) = 2 \quad \tau(2) = 3 \qquad \tau = (23) \end{array} \right)$$

3. in $GL(n,K)$:

$G = U(n) := \{ A \in M_n(\mathbb{C}) \mid AA^\dagger = 1_n \}$

conj. are similarity transformations How to label conj. classes?

Spectral theorem ensures $u \in U(n)$ can be diagonalized as. $\exists\, g \in U(n)$

↖ finite dim.

$$g u g^{-1} = \text{diag}(\underline{z_1 \cdots z_n}) \quad (|z_i| = 1)$$

↪ conjugacy classes labeled by $(z_1 \cdots z_n)$? $\nearrow U(1)^n$

permutation $A(\phi)\, \text{diag}(\underline{z_1 \cdots z_n}) A(\phi)^{-1}$

$$= \text{diag}(\underline{z_{\phi(1)}, z_{\phi(2)} \cdots z_{\phi(n)}})$$

$$[A(\phi) g] u [A(\phi) g]^{-1} = \text{diag}\{ z_{\phi(i)} \}$$

$\Rightarrow$ $U(1)^n / S_n$ labels conj. class.

4. a general element of $GL(n,\mathbb{C})$ is not diagonalizable. Define the characteristic polynomial $(A \in GL(n))$

$$P_A(x) = \det(x\mathbb{1} - A)$$

$$P_{gAg^{-1}}(x) = \det(x\mathbb{1} - gAg^{-1})$$

$$= \det(g(x\mathbb{1} - A)g^{-1})$$

$$= \det(x\mathbb{1} - A) = P_A(x)$$

<u>Definition</u> A <u>class function</u> on a group is a function f on G, s.t.

$$f(g g_0 g^{-1}) = f(g_0) \qquad \forall g, g_0 \in G.$$

For a matrix representation. define the <u>character</u> of the representation

$$\chi_T(g) := \mathrm{Tr}\, T(g)$$

It is a class function.

<u>Definition</u>. Two homomorphisms $\varphi_i : G_1 \to G_2$ are conjugate if $\exists g_2 \in G_2$, s.t.

$$\varphi_2(g_1) = g_2 \varphi_1(g_1) g_2^{-1}$$

in terms of representations $(T: G \to GL(V))$

$$\begin{array}{ccc} V_1 & \xrightarrow{S} & V_2 \\ T_1(g)\downarrow & & \downarrow T_2(g) \\ V_1 & \xrightarrow{S} & V_2 \end{array}$$

(equivariant map / morphism of G-space)

$$T_2(g) S = S T_1(g) \qquad (\dim V_1 = \dim V_2)$$

$$T_2(g) = S T_1(g) S^{-1} \Leftarrow \text{equivalent representation}$$

5. Conjugacy classes in S_n. (§7.5 of Moore)

Permutations with same structure of cycle decomposition are conjugate.

The conjugacy classes are labeled by the cycle decomposition of their elements. $C(\vec{l})$

$\vec{l} = (l_1, l_2, \cdots, l_n)$ where l_r is the number of r-cycles.

$$n = \sum_{j=1}^{n} j \cdot l_j$$

$$\phi = (12)(34)(678)(11,12) \in S_{12}$$

$$= (12)(34)(5)(678)(9)(10)(11,12)$$

$$\vec{l} = \begin{matrix} l_1 & l_2 & l_3 & l_{\geq 4} \\ 3, & 3 & 1 & 0 \end{matrix} \qquad \vec{l} = (3, 3, 1, 0, \cdots 0)$$

$\Rightarrow$ The number of conjugacy classes of S_n is given by the partition function of n, $\underline{p(n)}$. namely the number of distinct partitions of n into sum of nonnegative integers.

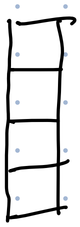
Example S_4

partition	cycle decomp.	tyical g	$\lvert C(g)\rvert$	order of g
$4 = 1+1+1+1$	$(1)^4$	1	1	1
$4 = 1+1+2$	$(1)^2(2)$	(ab)	$\binom{4}{2} = 6$	2
$4 = 1+3$	$(1)(3)$	(abc)	$2\binom{4}{3} = 8$	3
$4 = 2+2$	$(2)^2$	$(ab)(cd)$	$\frac{1}{2}\binom{4}{2} = 3$	2
$4 = 4$	(4)	$(abcd)$	6	4

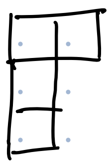
$|S_4| = 24 = 1+6+8+3+6$ $\qquad p(4) = 5$

Young diagram : $\quad n = \sum_{i=1}^{k} \lambda_i \quad \lambda_i \geq \lambda_{i+1} \geq 0$

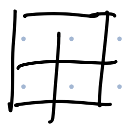
λ_1 λ_2 λ_3 λ_4



$(1)^4$



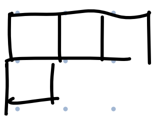
$(1)^2(2)$



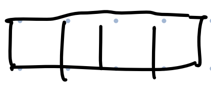
$(2)^2$

λ_i : number of boxes in i-th row

$\lambda_i \geq \lambda_{i+1}$.



$(1)(3)$



(4)

Define a partition : $\lambda = \{\lambda_1, \lambda_2 \cdots\}$ $\quad n \overset{k}{=} \sum_{i=1} \lambda_i$

multiplicity : $m_i(\lambda) = |\{ j \mid \lambda_j = i \}|$

label conj. class $C(\lambda) := (1)^{m_1}(2)^{m_2} \cdots$

– 6.3. Normal subgroups & Quotient groups

<u>Defition</u> A subgroup $N \subset G$ is called a normal subgroup or an invariant subgroup if

$$gNg^{-1} = N \quad \forall g \in G.$$

denoted $N \triangleleft G$. (Self-conjugate subgroups)

* NB. it doesn't mean $gng^{-1} = n \quad \forall n \in N$!

Suppose a subgroup Z satisfies

$$gzg^{-1} = z \quad \forall z \in Z \quad \forall g \in G.$$

$$Z(G) := \{ z \in G \mid zg = gz \quad \forall g \in G \}$$

$Z(G)$ is an abelian normal subgroup of G.

$Z(G)$ is the <u>center of G</u>.

<u>Examples</u>.

1. G is abelian. all subgroups are normal.

$$ghg^{-1} = (gg^{-1})h = h \quad \forall h \in G.$$

2. The kernel of a homomorphism

$$\phi : G \longrightarrow G'$$

is a normal subgroup.

$k \in \ker(\phi)$. $\phi(k) = 1_{G'}$

$$\phi(gkg^{-1}) = \phi(g)\cancel{\phi(k)}\phi(g^{-1}) = \phi(g)\phi(g)^{-1} = 1 \quad (\forall g \in G)$$

$\Rightarrow gkg^{-1} \in \ker(\phi)$

$\Rightarrow \ker\phi \triangleleft G$

Theorem. If $N \triangleleft G$, then the set of left cosets $G/N = \{gN, g \in G\}$ has a natural group structure with group multiplication defined as

- $(g_1 N)\cdot(g_2 N) := (g_1 g_2)N$

We call the groups of the form G/N quotient groups. (factor groups)

$$g_1 N \cdot g_2 N = g_1 (g_2 g_2^{-1}) N g_2 N$$

$$= g_1 g_2 \underbrace{(g_2^{-1} N g_2)}_{= N} N$$

$$= g_1 g_2 N$$