5.2. Cayley's theorem

Theorem (Cayley, 1878)

Every group G is isomorphic to a subgroup of S_G ("can be embeded in S_G)

In particular, if $|G| = n$, then G is isomorphic to a subgroup of S_n.

Proof: recall group action. let $X = G$. define left-mult.

$$\forall h. \quad L(h): G \longrightarrow G$$
$$g \longmapsto h \cdot g$$

$L(h) \in S_G$ as it is one-one and onto

and naturally $L(h_1) \cdot L(h_2) = L(h_1 h_2)$

So the map $L: h \longmapsto L(h)$ is a homomorphism.

L is one-one. thus $G \cong L(G) \subset S_G$

($S_G \cong S_N$ with an ordered set.

$\{1, \omega, \omega^2, \cdots \omega^{N-1}\} =: \mu_N \qquad S_{\mu_N} \cong S_N$

"natural ordering"

D_n, $SU(n)$. has no natural order)

Example. $\mathbb{Z}_n \cong \langle (1\,2 \cdots n) \rangle \cong \mu_n$

$n = 3$

$$\langle (123) \rangle = \{1, (123), (132)\} = A_3 \subset S_3$$

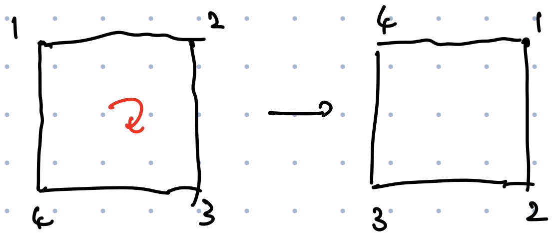
$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mu_3 \quad \{1, \quad \omega \quad \omega^2\}$$

Example. $D_4 = \langle A B \mid A^4 = B^2 = (AB)^2 = 1 \rangle$

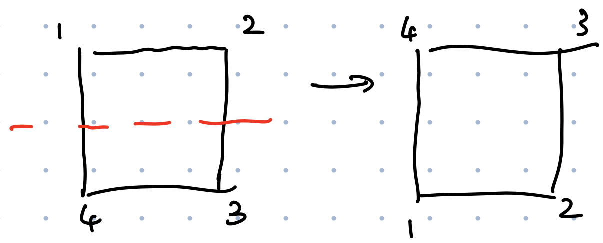
$|D_4| = 8 \cong$ a subgroup of S_8

A :



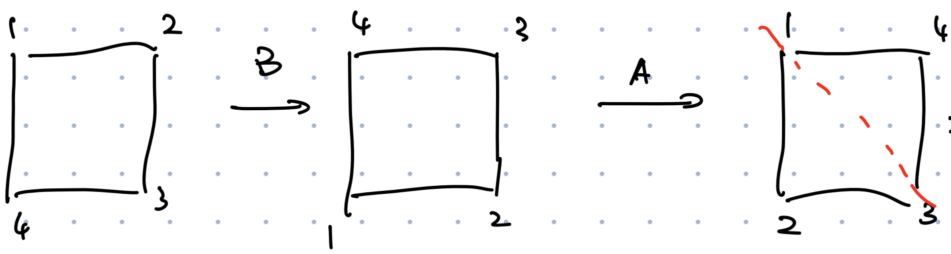
$:= (1\,2\,3\,4)$ subgroup of S_4

B :



$:= (14)(23)$

AB :



$:= (2\,4)$

$(AB)^2 = 1$

How to find the isomorphism?

→ use multiplication table (Cayley table)

Klein's 4-group. $V = \langle a\, b \mid a^2 = b^2 = (ab)^2 = e \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$e = (0, 0)$

$a = (1, 0)$

$b = (0, 1)$

$c = (1, 1)$

$|V| = 4$

$\phi : V \longrightarrow im(V) \subset S_4$

$a \longmapsto \phi(a)$

		e	a	b	c
1	e	e	a	b	c
2	a	a 2	e 1	c 4	b 3
3	b	b 3	c 4	e 1	a 2
4	c	c	b	a	e

$\phi(e) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$= 1$

$\phi(a) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$= (12)(34)$

$\phi(b) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

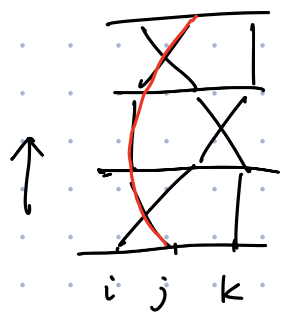
$= (13)(24)$

$\phi(c) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$

I.3. Transpositions / 2-cycles

i, j, k are distinct.

① $(ij)(jk)(ij) = (ik) = (jk)(ij)(jk)$



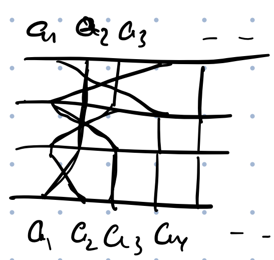
② $(ij)^2 = 1 \quad (ij) = (ij)^{-1}$

③ $(ij)(kl) = (kl)\cdot(ij) \quad \{i, j\} \cap \{k, l\} = \phi$

Theorem. Every permutation $\phi \in S_n$ is a product of transpositions.

Proof. $\phi \in S_n$ has a cycle decomposition.

For each cycle,



$(a_1 a_2 \cdots a_r) = (a_1 a_r)(a_1 a_{r-1}) \cdots (a_1 a_2)$

any permutation can be generated by transpositions

Remarks:

1. There are other ways to generate S_n

① $\sigma_i = (i \;\; i+1) \quad (1 \le i \le n-1)$

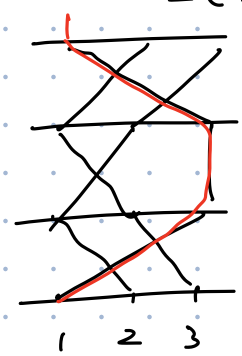
"elementary generators"

$(i\,j) = (i.\,i+1)(i+1.\,j)(i.\,i+1) \quad (i < j)$

② generated by $(1\,2)$ & $(1\,2 \cdots n)$

$(2\,3) = (1\,2 \cdots n)(1\,2)(1 \cdots n)^{-1}$

$:= (2\,3)$



Remark: transposition decomposition is not unique

$(1\,2\,3) = \underbrace{(1\,3)(1\,2)}_{2} = \underbrace{(2\,3)(1\,3)}_{2}$

$= \underbrace{(1\,3)(4\,2)(1\,2)(1\,4)}_{4}$

$= \underbrace{(1\,3)(4\,2)(1\,2)(1\,4)(2\,3)(2\,3)}_{6} - -$

always even number of transpositions

Definition A permutation $\phi \in S_n$ is even (odd) if it is a product of even (odd) transpositions. ("Parity")

(equivalent)

Definition If $\phi = \sigma_1 \cdots \sigma_t$ $\in S_n$ is a complete factorization into disjoint cycles (signum)

$$sgn(\phi) = (-1)^{n-t}$$

Cycle decomp. is unique $\Rightarrow$ sgn is well-defined

$(123) \in S_3$

- $sgn((123)) = (-1)^{3-1} = 1$ even.

- $S_6 \ni \phi = (123)(45) = (123)(45)(6)$ $n=6$, $t=3$

  $sgn = (-1)^3 = -1$

- transposition $\tau = (ij)$ $t = n-1$ $sgn(\tau) = (-1)^1 = -1$

- r-cycle. $t = (n-r)+1$

  $sgn\ \phi = (-1)^{n-t} = (-1)^{r+1}$

  r odd $\Leftrightarrow$ even perm.
  even $\Leftrightarrow$ odd.

- $sgn(\tau\phi) = sgn(\tau)\,sgn(\beta)$ actually

  $sgn(\alpha\beta) = sgn(\alpha)\,sgn(\beta)$

We can define a homomorphism:

$$\begin{aligned} sgn: \ & S_n \longrightarrow \mathbb{Z}_2 \\ & \phi \longmapsto sgn(\phi) \end{aligned}$$

Definition: The Alternating group $A_n \subset S_n$ is the subgroup of S_n of even permutations.

$$sgn(\phi) = 1, \quad \forall \phi \in A_n$$

① odd is a subgroup ?

② $A_2 = \{1\}$

$A_3 = \{1, (123), (132)\}$

$$A_4 = \{1, \left.\begin{matrix} (123), (132), \\ (124), (142) \\ (134)\ (143) \\ (234)\ (243) \end{matrix}\right\} 8$$

$(12)(34), (13)(24)$

$(14)(23) \}$

③ A_3 is Abelian $A_3 \cong \mathbb{Z}_3 \cong \mu_3$

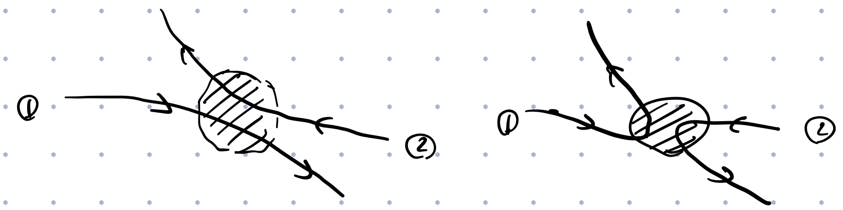
A_6 is not Abelian.

$$(123)(124) = (13)(24)$$

$$(124)(123) = (14)(23)$$

S_n and n identical particles

For identical particles. there is exchange degeneracy



final $|k_1\rangle|k_2\rangle$ $\qquad$ $|k_2\rangle|k_1\rangle$ $\qquad$ orth. if $k_1 \neq k_2$

$$\varphi = \alpha|k_1\rangle|k_2\rangle + \beta|k_2\rangle|k_1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

permutation P_{12} : $P_{12}|k_1\rangle|k_2\rangle = |k_2\rangle|k_1\rangle$

Then $P_{12}^2|k_1\rangle|k_2\rangle = |k_1\rangle|k_2\rangle$. $P_{12}^2 = \mathbb{1}$.

eigenspectrum ± 1.

Consider Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(|r_1 - r_2|) + V_{ext}(r_1) + V_{ext}(r_2)$$

$$P_{12} H P_{12}^{-1} = H \quad \rightarrow [P_{12}, H] = 0$$

have same eigenstates

construct symmetrizer / antisymmetrizer (projectors)

$$\hat{S} = \frac{1}{2}(\mathbb{1} + P_{12})$$

$$\hat{A} = \frac{1}{2}(\mathbb{1} - P_{12})$$

$\hat{S}^2 = \hat{S}$ $\qquad$ $\hat{S} + \hat{A} = \mathbb{1}$

$\hat{A}^2 = \hat{A}$ $\qquad$ $\hat{S}\hat{A} = \hat{A}\hat{S} = 0$

idempotent. (later in rep theory)

$$\begin{cases} \hat{P}_{12}\hat{S} = \frac{1}{2}(\hat{P}_{12} + \underbrace{\hat{P}_{12}^2}_{=\mathbb{1}}) = \hat{S} \\ \hat{P}_{12}\hat{A} = \frac{1}{2}(\hat{P}_{12} - \mathbb{1}) = -\hat{A} \end{cases}$$

given any state.

$$\begin{Bmatrix} \hat{S} \\ \hat{A} \end{Bmatrix} (\alpha|k_1\rangle|k_2\rangle + \beta|k_2\rangle|k_1\rangle) = \frac{1}{2}(\alpha|k_1\rangle|k_2\rangle + \beta|k_2\rangle|k_1\rangle)$$

$$\pm \frac{1}{2}(\alpha|k_2\rangle|k_1\rangle + \beta|k_1\rangle|k_2\rangle)$$

$$= \frac{\alpha \pm \beta}{2}(|k_1\rangle|k_2\rangle \pm |k_2\rangle|k_1\rangle)$$

For more particles n, it generalizes into

$$\hat{S} = \frac{1}{N!}\sum_\alpha \hat{P}_\alpha \qquad \hat{A} = \frac{1}{N!}\sum_\alpha \underbrace{\epsilon_\alpha}_{\mathrm{sgn}(P_\alpha)} \hat{P}_\alpha$$

symmetrization postulate

$$P_{ij}|\varphi_{boson}\rangle = |\varphi_{boson}\rangle$$

$$P_{ij}|\varphi_{fermion}\rangle = -|\varphi_{fermion}\rangle$$

($\hat{S}$. $\hat{A}$ construct 1D IR for S_n)