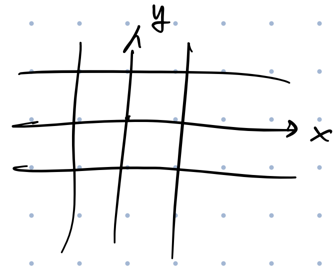
4. Group actions on sets (cont.)

<u>Example</u> Space group acts on a 2D square lattice

$p4mm = \{ \{ g | \vec{\tau} \} : g \in D_4 , \vec{\tau} \in a\hat{x} + b\hat{y}$
$(a, b \in \mathbb{Z}) \}$



group presentations ?

$\langle R, m_x, T_x, T_y \rangle$

Consider Wyckoff positions : orbits in real space.

1a. $(0,0)$ $4mm$

1b $(\frac{1}{2}, \frac{1}{2})$ $4mm$

2c $(\frac{1}{2}, 0)$ $(0, \frac{1}{2})$ m

4d $(\pm x, 0)$ $(0, \pm x)$ m

4e $(\pm x, \frac{1}{2})$ $(\frac{1}{2}, \pm x)$ m

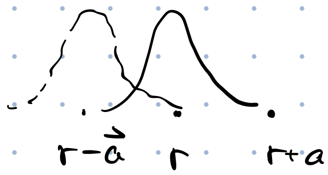
4f $(\pm x, \pm x)$ m

8g (x, y) 1

Consider wave function $\varphi(\vec{r})$: $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{C}$

How to define sym. ops. on φ ?

$$g \cdot \varphi(\vec{r}) = \varphi(g^{-1} \cdot \vec{r})$$



$$\varphi(T_x \vec{r}) = (T_x^{-1} \cdot \varphi)(\vec{r})$$

$$\Downarrow$$

$$(T_x \varphi)(\vec{r}) = \varphi(\vec{r} - \vec{a})$$

$$(\, g \cdot \varphi(g \cdot \vec{r}) = \varphi(\vec{r})\,)$$

4.2 Induced group actions on associated function spaces.

$\mathcal{F}[X \to Y]$ is the set of functions from set X to set Y. Let ϕ be the left group action

$$\phi : G \times X \to X.$$

Then there is also a group action $\tilde{\phi}$ on $\mathcal{F}$:

$$\tilde{\phi}(g, F)(x) := F(\phi(g^{-1}, x)) \qquad F \in \mathcal{F}.$$

$$\begin{aligned}
\tilde{\phi}(g_1, \tilde{\phi}(g_2, F))(x) &= \tilde{\phi}(g_2, F)(\phi(g_1^{-1}, x)) \\
&= F(\phi(g_2^{-1}, \phi(g_1^{-1}, x))) \\
&= F(\phi((g_1 g_2)^{-1}, x)) \\
&= \tilde{\phi}(g_1 g_2, F)(x)
\end{aligned}$$

$$(g \cdot F)(x) := F(g^{-1} \cdot x)$$

4.3 equivariant maps

<u>Definition</u> Let X, X' be two G-spaces

A <u>equivariant map</u>, $f: X \to X'$

satisfies

$$f(g \cdot x) = g \cdot f(x) \quad \forall x \in X \ \forall g \in G.$$

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \Phi(g) \downarrow & & \downarrow \Phi'(g) \\ X & \xrightarrow{f} & X' \end{array}$$

$$f(\Phi(g.x)) = \Phi'(g. f(x))$$

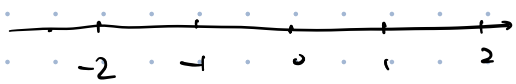
f is also called a morphism of G-spaces.

<u>Examples</u>.

$G = \mathbb{Z}$ acts on $\mathbb{R}$

$n : x \mapsto x + n$

orbits?



$\mathbb{R}/\mathbb{Z} = [0,1) \sim S^1$

- equivariant map?

$f: \mathbb{R} \to \mathbb{R}$

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\phi_n"'] & R \arrow[d, "\phi_n"] \\ R \arrow[r, "f"] & R \end{tikzcd}$$

$f(x) + n_1 = f(x + n_1)$

$f(x) + n_2 = f(x + n_2)$

$f(\underline{x} + n_1) - f(\underline{\ }x + n_2) = \underline{n_1 - n_2}$ $\quad \forall x, n_i$

$f(x) = x + \alpha$

5. The symmetric group

Recall that

Given a set X, the set of permutations

$$S_X := \{ X \xrightarrow{f} X \; : \; f : 1\text{-}1 \;\&\; \text{onto (invertible)} \}$$

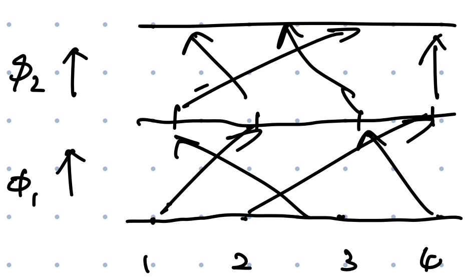
For $n \in \mathbb{N}^+$ denote the symmetric group on n elements, S_n, which is the set of all permutations of the set $X = \{1, 2, \cdots, n\}$

$(|S_n| = n!)$

A permutation can be written as

$$\phi = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix} \quad \text{with } p_i = \phi(i)$$

$$\phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$



$$\phi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$\phi_2 \cdot \phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

$$\phi_2 \cdot \phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

循环表示

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (2\ 4)$$

Canonical permutation rep. of S_n

"regular rep. of S_n" by Zee book. 正则表示

Consider S_n and an n-dim. carrier space ($\mathbb{R}^n$, $\mathbb{C}^n$ etc.)

with an ordered basis $\vec{e}_i = \{0, 0, \cdots 0, \underset{i\text{-th}}{\underline{1}}, 0, \cdots 0\}$

$\phi \in S_n$: $T(\phi)$: $\vec{e}_i \longrightarrow \vec{e}_{\phi(i)}$

$$T(\phi)\vec{e}_i = \sum_{j=1}^{n} A(\phi)_{ji}\, \vec{e}_j \qquad A \in GL(n, K)$$

$\left(A_{ji}(\phi) = \langle e_j | e_{\phi(i)} \rangle \right)$ $\qquad A_{ij}(\phi) = \langle e_i | e_{\phi(i)} \rangle$

$$\phi = (1234) \in S_4 \qquad A(\phi) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\phi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \qquad A(\phi_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\phi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \qquad A(\phi_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5.1. Cycles & transpositions

Definition. Let $i_1, \ldots i_r$ be distinct integers between 1 and n.

If $\phi \in S_n$ fixes the remaining integers and if

$$\phi(i_1) = i_2, \quad \phi(i_2) = i_3, \quad \ldots \quad \phi(i_r) = i_1$$

then ϕ is an r-cycle (cycle of length r)

$$(i_1\ i_2\ i_3 \cdots i_r)$$

A 2-cycle is called a transposition.

Remarks:

1. cycles are the same up to cyclic ordering

   $(234) = (423) = (342)$

2. disjoint cycles commute

   $(234)(56) = (56)(236)$

   $(12)(23) \neq (23)(12)$

3. inverse of a permutation

   $[(12)(345)]^{-1} = (12)(543) = (12)(354)$

Theorem: Every permutation $\phi \in S_n$ is either a cycle or can be factorized into disjoint cycles.

(Proof by induction)

(Def) complete factorization: is a product of disjoint cycles which contains one 1-cycle for each fixed x.

$$\underline{(1)(234)} \quad (\neq (1)(1)(1)(234))$$

Complete factorization of a permutation ϕ is unique (up to ordering). which we call the cycle decomposition of ϕ.