Recap: Homomorphism & isomorphism

$\varphi: G \longrightarrow G'$

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \varphi\times\varphi \downarrow & & \downarrow \varphi \\ G' \times G' & \xrightarrow{m'} & G' \end{array} \qquad \varphi(g_1 \cdot g_2) = \varphi(g_1)\cdot\varphi(g_2)$$

$\ker \varphi = \{ g \in G : \varphi(g) = 1_H \} \subset$

$\mathrm{im}\ \varphi = \varphi(G) \subset H$

isomorphism 同构: $\ker\varphi = \{1\}$ injective
$\mathrm{im}\ \varphi = H$ surjective

$SU(2) \longrightarrow SO(3)$

① isomorphism between vec. spaces. $\mathbb{R}^3 \longrightarrow H_2^0$ traceless herm. mat 2×2

$$\vec{x} \longmapsto \vec{x}\cdot\vec{\sigma} = \sum_i x_i \sigma^i$$

$$= \begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix} \in H_2^0$$

② homo. defined as conjugation action by $u \in SU(2)$

$$C_u : H_2^0 \longrightarrow H_2^0$$
$$m \longmapsto u m u^{-1}$$

③
$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{R(u)} & \mathbb{R}^3 \\ h \downarrow & & \downarrow h \\ H_2^0 & \xrightarrow{C_u} & H_2^0 \end{array}$$

$h \cdot R(u) = C_u \cdot h$

$(R(u)\cdot\vec{x})\cdot\vec{\sigma} = u(\vec{x}\cdot\vec{\sigma})u^{-1}$

We can show that $R(u) \in SO(3)$.

$\forall u \in SU(2)$. $R(u) = R(-u)$

R: $2 \to 1$ surjective map.

$$u = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$$

$$u \cdot \sigma_1 u^{-1} = \begin{pmatrix} -(\alpha\beta + \bar{\alpha}\bar{\beta}) & \alpha^2 - \bar{\beta}^2 \\ \bar{\alpha}^2 - \beta^2 & \alpha\beta + \bar{\alpha}\bar{\beta} \end{pmatrix}$$

$$= (a^2 - b^2 - c^2 + d^2)\sigma_1 + (-2ab - 2cd)\sigma_2 + (-2ac + 2bd)\sigma_3$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{R(u)} \begin{pmatrix} a^2 - b^2 - c^2 + d^2 \\ -2ab - 2cd \\ -2ac + 2bd \end{pmatrix}$$

$\alpha = a + ib$

$\beta = c + id$

$(a^2 + b^2 + c^2 + d^2 = 1)$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{R(u)} ?$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{R(u)} ?$$

3. Homomorphism & isomorphism (cont.)

<u>Example</u>. $GL(V)$ and $GL(n, K)$

Let $GL(V): V \to V$ be the group of invertible linear transformations with a finite dimensional vector space V.

Given an ordered basis $b = \{\hat{e}_1, \cdots \hat{e}_n\}$

Define a homomorphism:

$$\varphi_b : GL(V) \longrightarrow GL(n, K)$$

$$\tau \longmapsto T_b(\tau)$$

s.t. $\underline{\tau(\hat{e}_i) = \sum_j \hat{e}_j \cdot T_b(\tau)_{ji}}$ ⓐ

$\forall \vec{v} \in V$. $\vec{v} = \sum_{i=1}^{n} v_i \hat{e}_i \quad (v_i \in K)$

$$\tau \vec{v} = \sum_{i=1}^{n} v_i (\tau \hat{e}_i) = \sum_{ij} \hat{e}_j \cdot T_b(\tau)_{ji} v_i$$

$$\Rightarrow \tau_1(\tau_2 \vec{v}) = \sum_{ij} (\tau_1 \hat{e}_j) T_b(\tau_2)_{ji} v_i$$

$$= \sum_{ijk} \hat{e}_k \cdot T_b(\tau_1)_{k\underline{j}} T_b(\tau_2)_{\underline{j}i} v_i$$

$$= \sum_{ik} \hat{e}_k \underline{[T_b(\tau_1) T_b(\tau_2)]_{ki}} v_i$$

$$= (\tau_1 \tau_2) \vec{v}$$

$$= \sum_{ik} \hat{e}_k \underline{[T_b(\tau_1 \tau_2)]_{ki}} v_i$$

$$\Rightarrow T_b(\tau_1 \tau_2) = T_b(\tau_1) T_b(\tau_2)$$

$\left\{\begin{array}{l}\text{surjective } \checkmark \\ \text{injective ? } \quad \tau(\hat{e}_i) = e_i \Leftrightarrow \tau = id \quad \checkmark \end{array}\right.$

$$\Updownarrow$$

$$T_\ell(\tau) = \mathbb{1}_n$$

<u>isomorphism</u> $\underline{GL(V)} \cong \underline{GL(n,k)}$

<u>Definition</u>.

① Let G be a group. then a finite dimensional <u>representation</u> of G is a finite dimensional vector space V with a group homomorphism

$\varphi : G \longrightarrow GL(V)$

V: carrier space

② A matrix representation of G is a homomorphism

$$\varphi : G \longrightarrow GL(n,k) \qquad (K = \mathbb{R}, \mathbb{C})$$

$$g \longmapsto \underline{P}(g)$$

$$\forall g_1, g_2 \in G : \quad P(g_1 g_2) = P(g_1) P(g_2)$$

① + an ordered basis $\longrightarrow$ ② $\left(GL(V) \cong GL(n,k) \right)$

Matrix rep. is basis dependent

$$\hat{e}_i = \sum_{j=1}^{n} S_{ji} \hat{e}'_j$$

$$P'(g) = S P(g) S^{-1}$$

Definition (equivalent representation) Γ, Γ' are n-dim reps of G

Γ, Γ' are equivalent ($\Gamma \cong \Gamma'$) if $\exists S \in GL(n,k)$

s.t. $\forall g \in G \quad \Gamma'(g) = S\,\Gamma(g)\,S^{-1}$

Example $\mathbb{Z}, \mathbb{R}, \mathbb{C} \ni a$

$$\Gamma(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

$$\Gamma(a)\Gamma(b) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix}$$

Example. $S_2 = \{e, \sigma\} \qquad \sigma^2 = e$

$$S_2 \cong \mu_2 \cong \mathbb{Z}_2$$

$$\Gamma(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Gamma(\sigma^2) = \Gamma(\sigma)\cdot\Gamma(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example $\mu_3 = \langle \omega \mid \omega^3 = 1 \rangle$

$$\Gamma(e) = 1_3$$

$$\Gamma(\omega) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Gamma(\omega^2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Example $D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$ ✓

$|D_6| = 8$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$C = AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Gamma(D_6) = \{ \pm \mathbb{1}, \pm A, \pm B, \pm C \}$

isomorphism : "faithful representation"

not faithful. $\Gamma(A) = \Gamma(B) = \mathbb{1}$

4. Group actions on sets

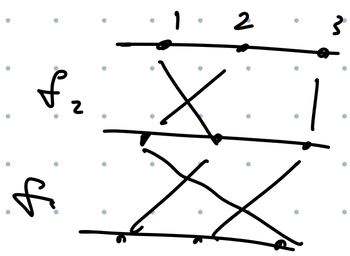
Definition: Given a set X, the set of permutations

$$S_X := \{ X \xrightarrow{f} X : f : 1-1 \ \& \ \text{onto (invertible)} \}$$

is a group under composition.

$$m(f_1, f_2) := f_1 \circ f_2$$

$$\begin{tikzcd} X \arrow[r, shift left, "f_2"] \arrow[rr, bend right, "f_1 \circ f_2"'] & X \arrow[l, shift left, "f_2^{-1}"] \arrow[r, shift left, "f_1"] & X \arrow[l, shift left, "f_1^{-1}"] \end{tikzcd}$$



Definition. A (left) group action Φ of G is a homomorphism

$$\Phi : G \longrightarrow S_X \qquad \phi(g, \cdot) : X \to X$$

$$g \longmapsto \underline{\phi(g, \cdot)} \qquad x \mapsto \phi(g, x)$$

1. $\phi : G \times X \longrightarrow X \qquad \phi(g, x) \in X \quad (\forall x \in X)$

$$\phi(g_1, \phi(g_2, x)) = \phi(\underline{g_1 g_2}, x)$$

$$\phi(1_G, x) = x \quad (\forall x \in X)$$

$$\phi(g, \phi(g^{-1}, x)) = \phi(g g^{-1}, x) = \phi(1_G, x) = x$$

simplified notation. $g \cdot x := \phi(g, x)$

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad (\forall x \in X)$$

Definition: If a set X has a group action by G we say that X is a **G-set**.

Example 1. $X = G$.

① group action by **multiplication**

$x \in X = G$

$g_1 \cdot (g_2 x) = g_1 g_2 x = (g_1 g_2) x$

② group action by **conjugation**

$g \cdot x := g x g^{-1} \in G = X$

a. $g_1 \cdot (g_2 \cdot x) = g_1 (g_2 x g_2^{-1}) = g_1 g_2 x g_2^{-1} g_1^{-1}$

$= (g_1 g_2) \cdot x$

b. $e \cdot x = e x e^{-1} = x$

Abelian group. $g \cdot x = g x g^{-1} = x \quad (\forall g \in G)$

2. $GL(n, K)$ acts on K^n.

$A \cdot \vec{v} = \sum_j A_{ij} v_j$

$e = 1_n$

a rep. of G $\Rightarrow$ group action on carrier space V.

3. Space group action on $\mathbb{R}^3$

$$\{g|\tau\} \qquad g \in O(3)$$

$$\tau \in T \quad \text{translation}$$

$$\{R_g|\vec{\tau}\}\cdot\underline{\vec{r}} := R_g\vec{r}+\vec{\tau} \qquad R_g \in O(3)$$

$$\underline{\{R_1|\vec{\tau}_1\}\{R_2|\vec{\tau}_2\}\cdot\vec{r}} = \{R_1|\vec{\tau}_1\}(R_2\vec{r}+\vec{\tau}_2)$$

$$= R_1(R_2\vec{r}+\vec{\tau}_2)+\vec{\tau}_1$$

$$= \underline{\{R_1R_2\,|\,R_1\vec{\tau}_2+\vec{\tau}_1\}\cdot\vec{r}}$$

matrix rep.

$$\{g|\vec{\tau}\} = \begin{array}{c} \left(\begin{array}{c|c} 1 & 0 \\ \hline \vec{\tau} & R_g \end{array}\right) \end{array} \begin{array}{c} 1 \\ 3 \end{array} \qquad \vec{r} \to \begin{pmatrix} 1 \\ \vec{r} \end{pmatrix}$$

$$\{R_1|\vec{\tau}_1\}\{R_2|\vec{\tau}_2\} = \begin{pmatrix} 1 & 0 \\ \tau_1 & R_1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \tau_2 & R_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ R_1\tau_2+\tau_1 & R_1R_2 \end{pmatrix}$$

$$= \{R_1R_2\,|\,R_1\vec{\tau}_2+\vec{\tau}_1\}$$

<u>Definition</u> (Orbits). Let X be a G-set

the orbit of G through a point $x \in X$ is

the set

$$O_G(x) := \{ g \cdot x \mid \forall g \in G \}$$

$$= \{ y \in X : \exists g, \text{ s.t. } y = g \cdot x \}$$

This defines an equivalence relation "$\sim$"

($x \sim x$; $x \sim y \Leftrightarrow y \sim x$; $x \sim y$, $y \sim z \Rightarrow x \sim z$)

$O_G(x)$ are equivalence classes ($[x]$) under

group action.

Distinct orbits of G partition X:

① $\forall x (\in X) \in O_G(x)$

② If $O_G(x_1) \cap O_G(x_2) \ni x \Rightarrow O_G(x_1) = O_G(x_2)$

($x = g_1 \cdot x_1 = g_2 \cdot x_2$ $\quad \underline{x_1} = g_1^{-1} \cdot g_2 \cdot \underline{x_2}$)

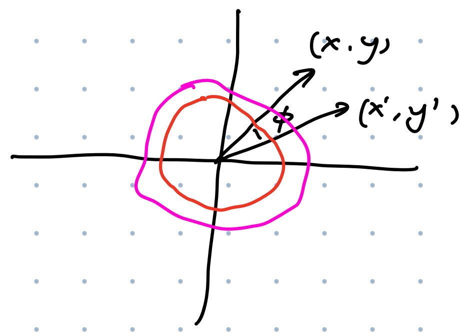
$\Rightarrow$ X is covered by disjoint orbits.

The set of orbits is denoted as X/G

Examples

1. $G = SO(2, \mathbb{R})$. on $\mathbb{R}^2$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$



$$\mathbb{R}^2 / SO(2) = [0, +\infty)$$

2. $G = C_3 = \{ R(0), R(2\pi/3), R(4\pi/3) \} \cong \mathbb{Z}_3$
$\subset SO(2, \mathbb{R})$

X : eq las. triangle