1. Introduction

Group $\Longleftrightarrow$ symmetry $\Longleftrightarrow$ conservation

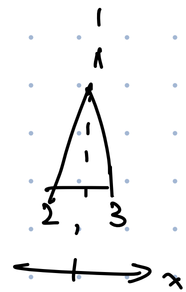
量子化. 对称性. 相因子

h Minkowski: $O(1,d)$

Weyl : Theory of group and
Wigner: Quantum Mechanics
(1928)
(Wigner theorem)

① discrete symmetry.

reflection. $m: x \longrightarrow -x$



$$m \overset{1}{\underset{2\;3}{\Delta}} = \overset{1}{\underset{3\;2}{\Delta}}$$

$$m^2 \overset{1}{\underset{2\;3}{\Delta}} = \overset{1}{\underset{2\;3}{\Delta}} \qquad m^2 = E$$

$G_1 = \{E, m\}$
↑ identity

particle permutation. $|\varphi(\vec{r}_1, \vec{r}_2)\rangle$

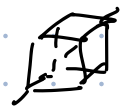
$$P\,|\varphi(\vec{r}_1, \vec{r}_2)\rangle = |\varphi(\vec{r}_2, \vec{r}_1)\rangle$$

$$P^2\,|\varphi(\vec{r}_1, \vec{r}_2)\rangle = |\varphi(\vec{r}_1, \vec{r}_2)\rangle$$

$G_2 = \{E, P\}$ $\quad P|\psi\rangle = \pm|\psi\rangle$ $\begin{cases} 1 & \text{boson ②} \\ -1 & \text{fermion} \end{cases}$

$G_1 \cong G_2 \cong \mathbb{Z}_2 = \{1, -1\}_\times \cong S_2$

<u>Condensed matter</u>:



Point groups (32)
\+ translation symmetry. $\Big\}$ → space groups (230)

↓

- energy levels → transition selection rules
- lattice vibrations / phonons

→ phonon spectra

⇓

Infra-red spectroscopy
{ Raman

② continuous symmetry.

time-translation ⟺ energy conservation

spatial-translation ⟺ momentum

rotational sym. ⟺ angular momentum

Lie group : $SU(2) \leftrightarrow SO(3)$

$\uparrow$
spin

standard model. $\underbrace{U(1) \times SU(2)} \times \overline{SU(3)}$

2. Groups : Basic definitions & examples

Def. A group is a quartet $(G, \underline{m}, \underline{I}, e)$

1. G is a set
2. $\underline{m}$: "multiplication" $G \times G \to G$.
3. $\underline{I}$: "inversion" $G \to G$
4. $e \in G$ identity element

They satisfy the following conditions:

1. (associativity):

$$\underline{m}(\underline{m}(g_1, g_2), g_3) = \underline{m}(g_1, \underline{m}(g_2, g_3))$$

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

Counter examples.

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

octonians (八元数)

2. (existence of id.) $\exists e$ s.t. $\forall g \in G$

$$e \cdot g = g \cdot e = g$$

3. (existence of inv.) $\forall g \in G$. $\exists I(g) =: g^{-1} \in G$

$$g \cdot g^{-1} = g^{-1} \cdot g = \underline{e}$$

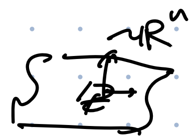
Remarks.

1. $e = 1 = 1_G = 0$ different notations for e

2. ① G a set, ② $\times$ associative : semigroup

   \+ ③ $\exists e$: monoid.

   \+ ④ $\exists g^{-1}$ group

3. G is a manifold 

   $\underline{m}$. $\underline{I}$. real analytic in local coordinates

   Lie group

4. $(G, \underline{m}, \underline{I}, e) =: G$

Q: a. is "e" unique ?

   b. g^{-1} unique (HW)

Examples

:

1. $G = \mathbb{Z}, \mathbb{R}, \text{ or } \mathbb{C}.$

$\underline{m}(a,b) := a+b \qquad a,b \in G$

$\begin{cases} e: 0 \\ I: - \end{cases}$

2. $\underline{m}(a,b) := ab$?

$G = \mathbb{R}^* := \mathbb{R} - \{0\}$ ✓

$\mathbb{C}^* := \mathbb{C} - \{0\}$ ✓

$\mathbb{Z}^* = \mathbb{Z} - \{0\}$ ×

Definition (subgroup)

(G, m, I, e) is a group. set $H \subset G$

m, I preserve H i.e. $m: H \times H \to H$

$I: H \to H$

(H, m, I, e) is a subgroup of $(G, --)$

if $H \neq G$: proper subgroup

Q.

1. $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$: m: "+" subgroups ✓

2. $\mathbb{R}^* \subset \mathbb{C}^*$ $\quad \mathbb{Z}^* \not\subset \mathbb{R}^*$

3. $\mathbb{R}^*$: $\mathbb{R}_{<0}$ $\bullet$ $\mathbb{R}_{>0}$ $\qquad \mathbb{R}_{>0} \subset \mathbb{R}^*$

4. H_1, H_2 are subgroups of G

   (a) $H_1 \cap H_2$ is a subgroup ? (HW)

   (b) $H_1 \cup H_2$

Definition (order of a group) $|G|$ is the cardinality of set G.

finite group if $|G| < \infty$, otherwise infinite group.

<u>Example</u>. the group of Nth roots of unity

$$\mu_N = \{ z \in \mathbb{C} \mid z^N = 1 \}$$

$$\omega = \exp\left(\frac{2\pi i}{N}\right)$$

$$\mu_N = \{ 1, \omega, \cdots \omega^{N-1} \}$$

$$\omega^i \cdot \omega^j = e^{i\frac{2\pi}{N}(i+j \bmod N)} = \omega^k$$

$$k = \underbrace{i+j \bmod N}_{\substack{!! \\ m}}$$

$$\mu_N \overset{u}{\cong} \mathbb{Z}_N = \{ 0, 1, \cdots N-1, \}$$

<u>Definition</u> (equivalence relation) " $\sim$ " is a binary relation. s.t. $\forall$ a.b.c $\in$ a set X

(1) $a \sim a$ reflexive

(2) $a \sim b \Leftrightarrow b \sim a$ symmetric

(3) $a \sim b$. $b \sim c \Rightarrow a \sim c$ transitive

An equivalence class of X is a subset

$$[a] := \{ x \in X \mid x \sim a \} \subset X$$

Example: residue classes modulo N

$0 \leq j \leq N-1$ $\quad [j] = \{ n \in \mathbb{Z} \mid j = n \bmod N \}$

$(\bar{j})$

$\underline{m}(\bar{r_1}, \bar{r_2}) := \overline{r_1 + r_2}$

$\mathbb{Z}_N$ or $\mathbb{Z}/N\mathbb{Z}$ " $/\sim$ "

$\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ $\qquad [0] = 0, 2, 4 \cdots$

$[1] = 1, 3, \sim$

$\bar{0} + \bar{0} = \bar{0}$

$\bar{0} + \bar{1} = \bar{1}$

$\bar{1} + \bar{1} = 0$

$\mathbb{Z}_2 = \{-1, 1\}$ $\qquad \underline{m} : i \cdot j$

<u>Definition</u> (direct product of groups) $G_1 \times G_2$

$n_1 \quad n_2$

$$(g_1, g_2) \in G_1 \times G_2$$

$$m_{G_1 \times G_2}((g_1, g_2), (g_1', g_2'))$$

$$:= (m_{G_1}(g_1, g_1'), m_{G_2}(g_2, g_2'))$$

$$(g_1, g_1' \in G_1, \quad g_2, g_2' \in G_2)$$

<u>Example.</u> $G_1 = G_2 = \mathbb{Z}_2 = \{1, \underline{-1}\}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$I = (1, 1)$$

$$a_1 = (-1, 1)$$

$$a_2 = (1, -1)$$

$$a_3 = (-1, -1)$$

So far. $m(a, b) = m(b, a) \qquad a \cdot b = b \cdot a$

Definition $\forall a, b \in G$. $a \cdot b = b \cdot a$ Abelian group

$\exists a, b \in G$, s.t. $a \cdot b \neq b \cdot a$ non-Abelian

for Abelian groups. $m(a, b)$ written as $a + b$

$e = 0$

Example (The general linear group)

$M_n(K)$. $n \times n$ matrices defined on field K

$(K = \mathbb{R}, \mathbb{C})$

$GL(n, K) := \{ A \in M_n(K) \mid \det A \neq 0 \}$

$A \cdot B \neq BA \quad (n \geq 2)$

Definition (center of a group) $Z(G)$

$Z(G) := \{ z \in G \mid z \cdot g = g \cdot z. \ \forall g \in G \} < G$

$\hookrightarrow Z(G)$ is an <u>Abelian</u> subgroup of G.

$Z(GL(n, K)) = \{ \lambda \mathbb{1}_n . \ \lambda \in K^* \}$

Examples: standard matrix groups $< GL(n, K)$

1. special linear group

$SL(n, K) = \{ A \in GL(n, K) \mid \underline{\det A = 1} \}$

2. orthogonal group & special O.G.

$$O(n,K) = \{ A \in GL(n,K) \mid AA^T = \mathbb{1} \}$$

$$(\equiv A^T A = \mathbb{1})$$

$$\det A \cdot \det(A^T)$$
$$= (\det A)^2 = 1$$
$$\det A = \pm 1$$

$$SO(n,K) = \{ A \in O(n,K) \mid \det A = 1 \}$$

3. (special) unitary group. on $\mathbb{C}$.

$$U(n) := \{ A \in GL(n,\mathbb{C}) \mid AA^{+} = \mathbb{1} \}$$

$$(\det A)(\det A)^* = 1$$
$$|\det A| = 1$$
$$\det A = e^{i\theta}$$

$$SU(n) = \{ A \in U(n) \mid \det A = 1 \}$$

4. indefinite orthogonal group

$$O(p,q) := \{ A \in GL(p+q, \mathbb{R}) \mid A^T J_{p,q} A = J_{p,q} \}$$

$$J_{p,q} = \mathrm{diag}\{ \underbrace{-1, -1, \cdots -1}_{p}, \underbrace{1, \cdots 1}_{q} \} = \left(\begin{array}{c|c} -\mathbb{1}_p & 0 \\ \hline 0 & \mathbb{1}_q \end{array} \right)$$

$$J_{1,3} = \mathrm{diag}\{-1, 1, 1, 1\} \qquad O(1,3)$$

5. symplectic group

$$Sp(2n, K) = \{ A \in GL(2n, K) \mid A^T J A = J \}$$

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \quad (J = J^* = -J^T = -J^{-1})$$

$\longrightarrow$ classical mechanics