

Problem 22 (Haar measure of $SU(2)$)¹

Recall that an element of $SU(2)$ $g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ can be expressed using Euler angles with $\alpha = e^{i\frac{1}{2}(\phi+\psi)} \cos \theta/2$ and $\beta = ie^{i\frac{1}{2}(\phi-\psi)} \sin \theta/2$.

(a) Show that

$$\int_{SU(2)} dg g_{\alpha\beta} = 0$$

$$\int_{SU(2)} dg g_{\alpha\beta} g_{\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}.$$

ϵ here is the Levi-Civita / totally antisymmetric tensor. In 2D, $\epsilon = i\sigma_2$.

(b) Using left and right-invariance show that

$$\int_{SU(2)} dg g_{\alpha_1\beta_1} \cdots g_{\alpha_n\beta_n}$$

can be nonzero if n is even and half of the α 's (β 's) are 1.

Problem 23

Find three different real irreps for S_3 . (*Hint*: two are one dimensional and one two dimensional.)

Problem 24

Recall from the lecture the decomposition of the canonical representation of S_n , $\mathbb{R}^n = W \oplus W^\perp$, where W is the one-dimensional invariant subspace spanned by the vector $v = \sum_{i=1}^n \hat{e}_i$ and W^\perp is the orthogonal complement of W . Show that W^\perp is irreducible.

¹pp.236-237 of [GM]