①

P 30. $D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = 1 \rangle$

$= \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$

$(\overset{\shortparallel}{sr^3} \quad \overset{\shortparallel}{sr^2} \quad \overset{\shortparallel}{sr})$

(1) $D_4 = \{1\} \cup \{r, r^3\} \cup \{r^2\} \cup \{s, r^2s\} \cup \{rs, r^3s\}$

$=: [1] \cup [r] \cup [r^2] \cup [s] \cup [rs]$

$C_1 = 1$; $C_2 = r + r^3$; $C_3 = r^2$; $C_4 = s + r^2 s$; $C_5 = rs + r^3 s$

(2)

	C_1	C_2	C_3	C_4	C_5
C_1	C_1	C_2	C_3	C_4	C_5
C_2		$2C_1+2C_3$	C_2	$2C_5$	$2C_4$
C_3			C_1	C_4	C_5
C_4				$2C_1+2C_3$	$2C_2$
C_5					$2C_1+2C_3$

(3) $L_{jk} = \sum_i C_{ij}^{k}\, y^i$

$$L = \begin{pmatrix} y^1 & y^2 & y^3 & y^4 & y^5 \\ 2y^2 & y^1+y^3 & 2y^2 & 2y^5 & 2y^4 \\ y^3 & y^2 & y^1 & y^4 & y^5 \\ 2y^4 & 2y^5 & 2y^4 & y^1+y^3 & 2y^2 \\ 2y^5 & 2y^4 & 2y^5 & 2y^2 & y^1+y^3 \end{pmatrix}$$

$\lambda_a = y^1 \qquad - y^3$ $\qquad m_1 = 1$

$\lambda_b = y^1 + 2y^2 + y^3 - 2y^4 - 2y^5$ $\qquad m_2 = 2$

$\lambda_c = y^1 - 2y^2 + y^3 + 2y^4 - 2y^5$ $\qquad m_3 = 1$

$\lambda_d = y^1 - 2y^2 + y^3 - 2y^4 + 2y^5$ $\qquad m_4 = 2$

$\lambda_e = y^1 + 2y^2 + y^3 + 2y^4 + 2y^5$ $\qquad m_5 = 2$

$$\lambda_\mu = \frac{1}{n_\mu} \sum_{i=1}^{r} m_i \chi_\mu([C_i]) y^i \Rightarrow$$

$$\chi_a = n_a (1, 0, -1, 0, 0)$$

$$\chi_b = n_b (1, 1, 1, -1, -1)$$

$$\chi_c = n_c (1, -1, 1, 1, -1)$$

$$\chi_d = n_d (1, -1, 1, -1, 1)$$

$$\chi_e = n_e (1, 1, 1, 1, 1)$$

$$n_\mu = \left[\frac{|G|}{\sum_{i=1}^{r} m_i \left| \frac{\chi_\mu([C_i])}{n_\mu} \right|^2} \right]^{\frac{1}{2}} \Rightarrow n_a = 2, \quad n_b = n_c = n_d = n_e = 1$$

identify $[r] = C_4(z)$, $[r^2] = C_2(z)$, $[s] = C_2'$, $[rs] = C_2''$

We recover the character table.

Character table for point group D_4

D_4	E	$2C_4$ (z)	C_2 (z)	$2C'_2$	$2C''_2$	linear functions, rotations	quadratic functions	cubic functions
A_1	+1	+1	+1	+1	+1	-	x^2+y^2, z^2	-
A_2	+1	+1	+1	-1	-1	z, R_z	-	z^3, $z(x^2+y^2)$
B_1	+1	-1	+1	+1	-1	-	x^2-y^2	xyz
B_2	+1	-1	+1	-1	+1	-	xy	$z(x^2-y^2)$
E	+2	0	-2	0	0	(x, y) (R_x, R_y)	(xz, yz)	(xz^2, yz^2) (xy^2, x^2y) (x^3, y^3)

http://symmetry.jacobs-university.de/cgi-bin/group.cgi?group=304&option=4