P25. ① $G = (\mathbb{R}^+, \times)$

$\hat{G} = \mathrm{Hom}(G, U(1))$, then $\forall a \in G$, we have

$$\chi(a) = e^{ikf(a)} \in U(1), \ (k \in \mathbb{R} \ \ f: \mathbb{R} \to \mathbb{R})$$

$$\chi(a \cdot_G b) = \chi(ab) \equiv \chi(a) \cdot \chi(b)$$

$$\Leftrightarrow e^{ikf(ab)} = e^{ikf(a)} \cdot e^{ikf(b)}$$

$$\Leftrightarrow f(ab) = f(a) + f(b)$$

$$f(a) \propto \ln a$$

Let $\chi_k(a) = e^{ik \ln a}$

$$(\chi_{k_1} \cdot \chi_{k_2})(a) = \chi_{k_1}(a) \cdot \chi_{k_2}(a) = \chi_{k_1+k_2}(a) \qquad \forall k_1, k_2 \in \mathbb{R}$$

$$\Rightarrow \hat{G} = (\mathbb{R}, +), \text{ and } \hat{\hat{G}} = \widehat{(\mathbb{R}, +)} = (\mathbb{R}, +)$$

② $(\mathbb{R}, +) \cong (\mathbb{R}^+, \times)$

$$\varphi: (\mathbb{R}, +) \longrightarrow (\mathbb{R}^+, \times)$$

$$a \longmapsto e^a$$

It is easy to check that it is an isomorphism

$(e^a = 1 \text{ iff } a = 0)$

It follows from the Pontryagin – Van Kampen theorem, that is for locally compact abelian groups G: $\hat{\hat{G}} \cong G$.