P23 $\qquad g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$ ①

$$\alpha = e^{\frac{i}{2}(\phi+\varphi)} \cos\frac{\theta}{2} \qquad \beta = i e^{\frac{i}{2}(\phi-\varphi)} \sin\frac{\theta}{2}$$

$\phi \in [0, 2\pi)$ $\quad \theta \in [0, \pi)$, $\quad \varphi \in [0, 4\pi)$

take $\alpha = r e^{\frac{i}{2}(\phi+\varphi)} \cos\frac{\theta}{2} \Big|_{r=1}$ and similary for $\beta$.

$$d\alpha\, d\bar{\alpha}\, d\beta\, d\bar{\beta} = \left| \frac{\partial(\alpha, \bar{\alpha}, \beta, \bar{\beta})}{\partial(r, \varphi, \phi, \theta)} \right|_{r=1} d\varphi\, d\phi\, d\theta$$

$$= \left(\frac{1}{2} r^3 \sin\theta\right)\Big|_{r=1} d\varphi\, d\phi\, d\theta$$

$$= \frac{1}{2} \sin\theta \; d\varphi\, d\phi\, d\theta$$

since $|\det g| = 1$. $\quad g \to g_0 g$ does not contribute a factor that needs to be canceled.

normalization requires $\int C \cdot \sin\theta\, d\varphi\, d\phi\, d\theta = 1$

$$\Rightarrow \qquad C = \frac{1}{16\pi^2}$$

Haar measure $\quad \frac{1}{16\pi^2} \int d\varphi\, d\phi \sin\theta\, d\theta$

(b) $\phi_{\alpha\beta} \equiv \int dg\, g_{\alpha\beta} = \int dg\, (g_0 g)_{\alpha\beta} = g_{0\alpha\gamma} \int dg\, g_{\gamma\beta}$

$$g_0 \begin{pmatrix} \phi_{0\beta} \\ \phi_{1\beta} \end{pmatrix} = \begin{pmatrix} \phi_{0\beta} \\ \phi_{1\beta} \end{pmatrix} \quad (\forall g_0 \in SU(2))$$

choose $g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \phi_{0\beta} = \pm \phi_{1\beta} = 0$

$$\Rightarrow \underline{\int dg\, g_{\alpha\beta} = 0 \qquad \forall \alpha, \beta \in \{0, 1\}}$$

$(A^{\beta\delta})_{\alpha\gamma} = \int dg\, g_{\alpha\beta} g_{\gamma\delta} = \int dg\, (g_0 g)_{\alpha\beta} (g_0 g)_{\gamma\delta}$

$= (g_0)_{\alpha s} \int dg\, g_{s\beta}\, g_{t\delta}\, (g_0)_{\gamma t}$

$\Rightarrow A^{\beta\delta} = g_0 \cdot A^{\beta\delta} \cdot g_0^T \quad (\forall g_0 \in SU(2))$

$A^{\beta\delta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad$ take $g_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$$\Rightarrow A^{\beta\delta} = c_{\beta\delta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Similarly, $A^{\alpha\gamma} = c_{\alpha\gamma} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by right-invariance

$\Rightarrow A_{\alpha\gamma,\beta\delta} = c_{\beta\delta}\, \epsilon_{\alpha\gamma} = c_{\alpha\gamma} \cdot \epsilon_{\beta\delta}$

$\Rightarrow A_{\alpha\gamma,\beta\delta} = \underline{c \cdot \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}}$ $\qquad c = \frac{1}{2}$ by explicit calculation.

(c) $I = \int dg\, g_{\alpha_1\beta_1} \cdots g_{\alpha_n\beta_n}$

$$= \int dg\, (g_0 g)_{\alpha_1\beta_1} \cdots (g_0 g)_{\alpha_n\beta_n} \qquad (*)$$

1. $g_0 = -\mathbb{1}$. $(*) \Rightarrow I = (-1)^n I \Rightarrow I = 0$ for odd $n$.

2. $n$ even: $g_0 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ $(g_0)_{\alpha\beta} = \delta_{\alpha\beta}\, e^{(-1)^\alpha i\theta}$

$$(g_0 g)_{\alpha\beta} = e^{(-1)^\alpha i\theta} g_{\alpha\beta}$$

$$(*) \Rightarrow I = e^{i\theta \Sigma (-1)^{\alpha_i}} I \overset{I\neq 0}{\Rightarrow} \Sigma(-1)^{\alpha_i} = 0 \Rightarrow \text{half } \alpha_i\ 1,\ \text{half } \alpha_i\ 2$$

similarly, by right-invariance, half $\beta_i$ 1.

Option 2: explicit calculation

(b) ① $\int_{SU(2)} dg\, g_{ab} \propto \int_0^{4\pi} d\varphi\, e^{\pm i\varphi/2} = 0$

② $\int_{SU(2)} dg\, g_{ab} g_{cd} = \frac{1}{2} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta}$

show by explicit computation

zero terms contain phases $e^{\pm i\phi}$ ($\phi \in [0, 2\pi)$)

or $e^{\pm i\frac{\varphi}{2}}$ ($\varphi \in [0, 4\pi)$).

(c) $\int_{SU(2)} dg \; g_{\alpha_1\beta_1} \cdots g_{\alpha_n\beta_n}$ to be nonzero.

① n odd : must contain factor $e^{\pm i \frac{\phi}{2}} \Rightarrow 0$

② n even: each $\alpha$ should be paired with $\alpha^*$

i.e. $g_{11}$ paired with $g_{22}$

similar. $g_{12}$ paired with $g_{21}$.

$\Rightarrow$ half indices are 1 and the other half 2

P24 three irreps of $S_3$:

① trivial: $\rho(\phi) = 1 \quad \forall \phi \in S_3$

② sign-rep: $\rho(\phi) = \mathrm{sgn}(\phi)$.

③ $S_3 \cong D_3$. 2x2 rotation/reflection matrices

see lecture notes.